Calculus with Analytic Geometry II

Thomas R. Cameron

January 31, 2025

1 Inverse Trig Formulas

Recall the following derivative rules:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}},\ \frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}},\ \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Using integration by parts, we can use these derivative rules to find antiderivative rules for each inverse trig function. For example,

$$\begin{aligned}\int \arcsin(x)dx &= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx \qquad (u = \arcsin(x),\ dv = dx)\\ &= x\arcsin(x) + \frac{1}{2}\int \frac{du}{\sqrt{u}} \qquad \left(u = 1-x^2\right)\\ &= x\arcsin(x) + \sqrt{1-x^2} + C.\end{aligned}$$

2 Reduction Formulas

We can use integration by parts to produce reduction formulas for the integral of powers of sine or cosine. For instance, for $n \geq 2$, we have

$$\begin{aligned}\int \sin^n(x)dx &= \int \sin^{n-1}(x)\sin(x)dx\\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)dx\\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int (1-\sin^2(x))\sin^{n-2}(x)dx\\ &= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)dx - (n-1)\int \sin^n(x)dx\end{aligned}$$

Therefore, we have

$$\int \sin^n(x)dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx.$$

We can also derive reduction formulas for powers of tangent and secant. For instance, for $n \geq 2$, we have

$$\begin{aligned}\int \sec^n(x)dx &= \int \sec^{n-2}(x)\sec^2(x)dx \\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)dx \\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)(\sec^2(x)-1)dx \\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^n(x)dx + (n-2)\int \sec^{n-2}(x)dx\end{aligned}$$

Therefore, we have

$$\int \sec^n(x)dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx.$$

3 Exercises

I. Find the antiderivative of $\arctan(x)$.

II. Find the reduction formula for $\int \cos^n(x)dx$, where $n \geq 2$.

III. Find the reduction formula for $\int \tan^n(x)dx$, where $n \geq 2$.