Calculus with Analytic Geometry II

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1 Products of Sine and Cosine

If m and n are positive integers, then the integral

$$\int \sin^m(x)\cos^n(x)dx$$

can be evaluated using one of the following rules

n odd	• Split off a factor of $\cos(x)$ • Apply Pythagorean identity • Make substitution $u = \cos(x)$
m odd	• Split off a factor of $\sin(x)$ • Apply Pythagorean identity • Make substitution $u = \sin(x)$
m and n even	• Use Pythagorean identity to reduce integrand to only powers of cosine • Proceed using reduction formula for powers of cosine

Table 1: Integration rules for products of sine and cosine.

For example, consider the following integral

$$\begin{aligned}
\int \sin^4(x)\cos^5(x)dx &= \int \sin^4(x)\cos^4(x)\cos(x)dx \\
&= \int \sin^4(x)\left(1-\sin^2(x)\right)^2\cos(x)dx \\
&= \int u^4\left(1-u^2\right)^2 du \qquad (u = \sin(x)) \\
&= \int u^4\left(1-2u^2+u^4\right) du \\
&= \int \left(u^4-2u^6+u^8\right) du \\
&= \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C \\
&= \frac{1}{5}\sin^5(x) - \frac{2}{7}\sin^7(x) + \frac{1}{9}\sin^9(x) + C.
\end{aligned}$$

Note that if either m or n is odd, then the above strategy will work. For instance, in the following example the power of Cosine is odd but the power on Sine is $-1/2$:

$$\begin{aligned}
\int \frac{\cos^3(x)}{\sqrt{\sin(x)}}dx &= \int \frac{\cos^2(x)}{\sqrt{\sin(x)}}\cos(x)dx \\
&= \int \frac{(1-\sin^2(x))}{\sqrt{\sin(x)}}\cos(x)dx \\
&= \int \left(\sin^{-1/2}(x) - \sin^{3/2}(x)\right)\cos(x)dx \\
&= \int \left(u^{-1/2} - u^{3/2}\right)du \qquad (u = \sin(x)) \\
&= 2u^{1/2} - \frac{2}{5}u^{5/2} + C \\
&= 2\sin^{1/2}(x) - \frac{2}{5}\sin^{5/2}(x) + C.
\end{aligned}$$

Before proceeding, we note that integrals involving the products of sines and cosines of two angles can be simplified using the angle sum and difference identities.

2 Products of Tangent and Secant

If m and n are positive integers, then the integral

$$\int \tan^m(x)\sec^n(x)dx$$

can be evaluated using one of the following rules

n even	• Split off a factor of $\sec^2(x)$ • Apply Pythagorean identity • Make substitution $u = \tan(x)$
m odd	• Split off a factor of $\sec(x)\tan(x)$ • Apply Pythagorean identity • Make substitution $u = \sec(x)$
m even and n odd	• Use Pythagorean identity to reduce integrand to only powers of secant • Proceed using reduction formula for powers of secant

Table 2: Integration rules for products of tangent and secant.

3 Exercises

1. Evaluate the integral $\int \sin^4(x)\cos^4(x)dx$.
2. Evaluate the integral $\int \tan^2(x)\sec^4(x)dx$.
3. Evaluate the integral $\int \tan^2(x)\sec(x)dx$.