Calculus with Analytic Geometry II

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1 Trig Substitutions

We are concerned with integrals that contain radical expressions of the form

$$\sqrt{a^2 - x^2},\ \sqrt{x^2 + a^2},\ \sqrt{x^2 - a^2},$$

where $a > 0$. To this end, we make a trig substitution that allows us to apply the Pythagorean identity

$$a^2\cos^2(\theta) + a^2\sin^2(\theta) = a^2,$$

which can also be written as

$$a^2 + a^2\tan^2(\theta) = a^2\sec^2(\theta).$$

For example, consider the following integral

$$\begin{aligned}
\int \sqrt{a^2 - x^2}dx &= \int a\cos(\theta)\sqrt{a^2 - a^2\sin^2(\theta)}d\theta \qquad (x = a\sin(\theta),\ dx = a\cos(\theta)d\theta) \\
&= a^2\int \cos^2(\theta)d\theta \\
&= a^2\left(\frac{1}{2}\cos(\theta)\sin(\theta) + \frac{1}{2}\theta\right) + C \\
&= a^2\left(\frac{1}{2}\frac{\sqrt{a^2 - x^2}}{a}\frac{x}{a} + \frac{1}{2}\arcsin(x/a)\right) + C
\end{aligned}$$

As another example, consider the following integral

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4 - x^2}} &= \int \frac{2\cos(\theta)d\theta}{4\sin^2(\theta)\sqrt{4 - 4\sin^2(\theta)}} \qquad (x = 2\sin(\theta),\ dx = 2\cos(\theta)d\theta) \\
&= \frac{1}{4}\int \csc^2(\theta)d\theta \\
&= -\frac{1}{4}\cot(\theta) + C \\
&= -\frac{1}{4}\frac{\sqrt{4 - x^2}}{x} + C.
\end{aligned}$$

Often, we must complete the square before we can evaluate the integral using a trig substitution. For example, consider the following integral

$$\int \frac{dx}{\sqrt{x^2 - 4x + 8}} = \int \frac{dx}{\sqrt{(x-2)^2 + 4}} = \int \frac{du}{\sqrt{u^2 + 4}},$$

where the last equation is given by the u-substitution $u = x - 2$. Note that the last integral can be solved via the trig substitution $u = 2\tan(\theta)$.

2 Exercises

I. Find the arc length of the curve $y = x^2/2$ on the interval $[0, 1]$.

II. Find the area of the circle $x^2 + y^2 = r^2$, whre $r > 0$.

III. Evaluate the integral $\int \frac{\sqrt{x^2 - 25}}{x} dx$.