Calculus with Analytic Geometry II

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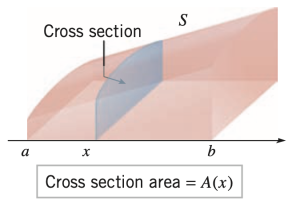
1 Volumes by Slicing

Let S denote a solid that extends along the x-axis and is bounded on the left and right, respectively, by planes that are perpendicular to the x-axis at $x = a$ and $x = b$. Let V denote the volume of S.



We can approximate V, if we know the area of each cross section, which we denote by $A(x)$. If we partition the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, $1 \leq i \leq n$, each of length Δx, then the volume of S can be approximated as

$$V \approx \sum_{i=1}^{n} A(c_i)\Delta x,$$

where c_i is any point in the subinterval $[x_{i-1}, x_i]$. Taking the limit as $n \to \infty$, our approximation becomes exact:

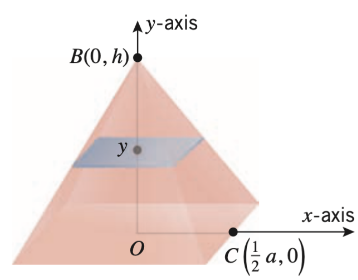
$$v = \lim_{n\to\infty} \sum_{i=1}^{n} A(c_i)\Delta x = \int_a^b A(x)dx.$$

As an example, consider the pyramid of height h and width a shown on the right. For any y in $[0, h]$ the cross section is given by a square. Let s denote the length of the square. Using similar triangles, we find that

$$s = \frac{a}{h}(h - y).$$

Therefore, the volume of the pyramid is given by

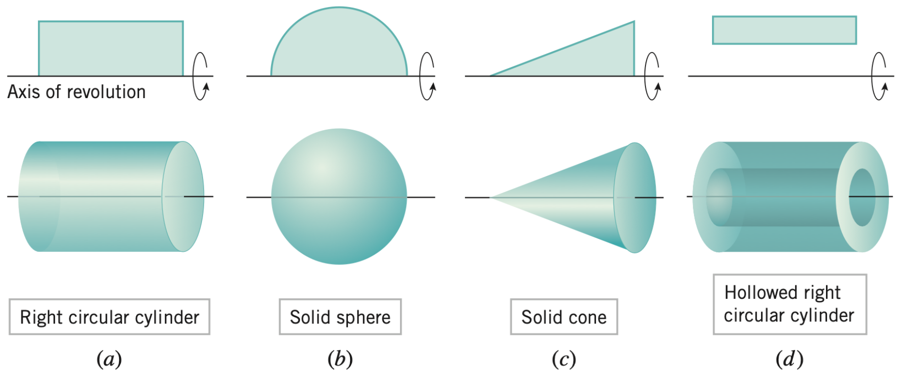
$$V = \int_0^h \frac{a^2}{h^2}(h - y)^2 dy = \frac{1}{3}a^2 h.$$



2 Solids of Revolution

A solid of revolution is formed by revolving a plane region about a line known as the axis of revolution, resulting in a solid whose cross sections are disks or washers of a certain radius.

We can determine the volume of a solid of revolution using the method of slicing. In particular, for disks the area of each cross section is given by $\pi f(x)^2$, where $f(x)$ denotes the radius of the disk. Note that $f(x)$ is the height of the original plane region measured from the axis of revolution. For washers, the area of each cross section is given by $\pi\left(f(x)^2 - g(x)^2\right)$, where $f(x)$ denotes the outer radius and $g(x)$ denotes the

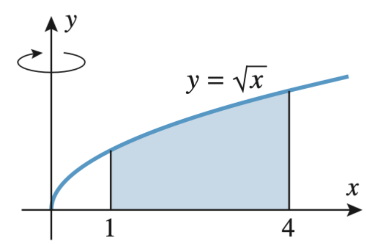


inner radius of the washer. Note that $f(x)$ and $g(x)$, respectively, denote the upper and lower bounds of the original plane region measured from the axis of revolution.

3 Shell Method

Sometimes the method of disks and washers is not the most efficient way to find the volume of a solid of revolution.

For example, consider the solid of revolution formed by taking the plane region bounded by $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x-axis, and revolving that region around the y-axis. Note that each cross section is a washer; however, the inner radius of the washer is $x = 1$, for $0 \leq y \leq 1$, and is $x = y^2$, for $1 \leq y \leq 2$. Hence, to find the volume of the solid of revolution using the washer method we need two integrals.



The shell method uses cyclindrical shells to approximate the volume of a solid of revolution. A cyclindrical shell is a solid enclosed by two circular cylinders. If the outer cylinder has radius r_2 and the inner cylinder has radius r_1, then the volume of the shell is given by

$$\begin{aligned} V &= \pi\left(r_2^2 - r_1^2\right)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi\left(\frac{r_1 + r_2}{2}\right)(r_2 - r_1)\,h. \end{aligned}$$

Now, divide the interval $[a, b]$ into n subintervals of length Δx. Then, the plane region has ben divided into n strips, which we denote by $R_1, R_2, \ldots, R_n$. When revolved around the y-axis, these strips generate tube-like solids $S_1, S_2, \ldots, S_n$ that are nested one inside the other, see Figure 1.

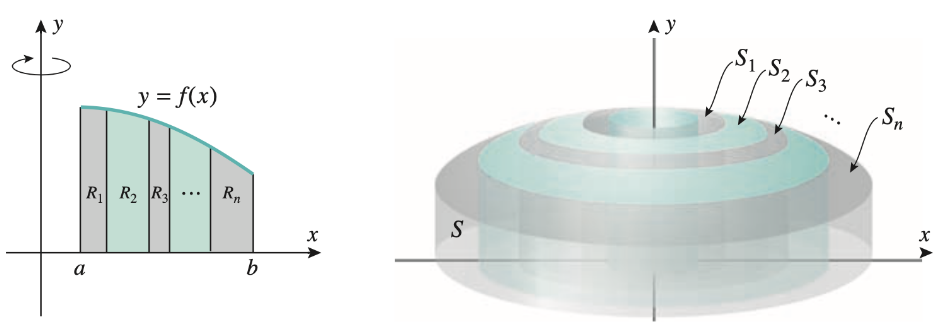


Figure 1: Shell Method, split the region into n strips and revolve around y-axis

Note that these strips have curved upper bounds. However, if the strips are thin we can approximate their areay using a rectangle. These rectangles, when revolved around the y-axis, will produce cylindrical shells whose volume closely approximate the volume of the tubes, see Figure 2.

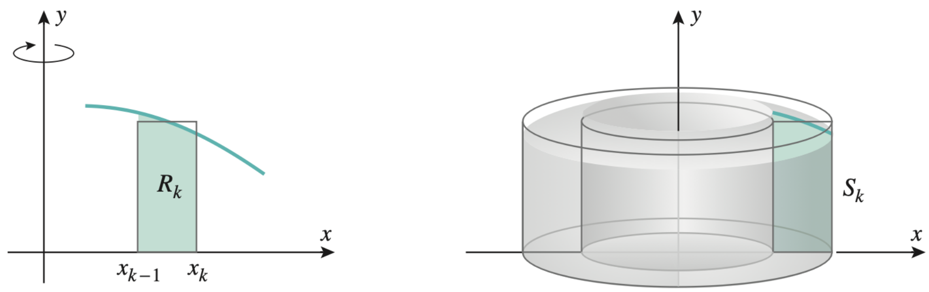


Figure 2: Shell Method, approximate the area of each strip by a rectangle and revolve around y-axis

To implement the shell method, consider the subinterval $[x_{k-1}, x_k]$ and let x_k^* denote the midpoint of this interval. If we construct a rectangle of height $f(x_k^*)$ over this interval, then revolving the rectangle around the y-axis will produce a cylindrical shell of average radius x_k^*, height $f(x_k^*)$, and thickness Δx. Hence, the volume of this shell is given by

$$V_k = 2\pi x_k^* f(x_k^*)\Delta x.$$

Therefore, we can approximate the volume of the solid of revolution as follows

$$V \approx 2\pi \sum_{i=1}^{n} x_i^* f(x_i^*)\Delta x.$$

Taking the limit as $n \to \infty$ gives us

$$V = 2\pi \lim_{n\to\infty} \sum_{i=1}^{n} x_i^* f(x_i^*)\Delta x = 2\pi \int_a^b x f(x) dx.$$