Calculus with Analytic Geometry II

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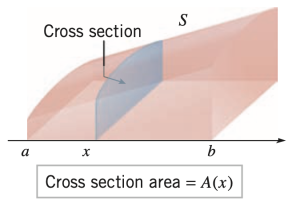
1 Volumes by Slicing

Let S denote a solid that extends along the x-axis and is bounded on the left and right, respectively, by planes that are perpendicular to the x-axis at $x = a$ and $x = b$. Let V denote the volume of S.



We can approximate V, if we know the area of each cross section, which we denote by $A(x)$. If we partition the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, $1 \leq i \leq n$, each of length Δx, then the volume of S can be approximated as

$$V \approx \sum_{i=1}^{n} A(c_i)\Delta x,$$

where c_i is any point in the subinterval $[x_{i-1}, x_i]$. Taking the limit as $n \to \infty$, our approximation becomes exact:

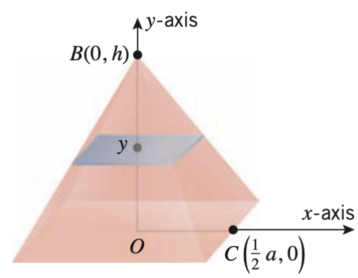
$$v = \lim_{n\to\infty} \sum_{i=1}^{n} A(c_i)\Delta x = \int_a^b A(x)dx.$$

As an example, consider the pyramid of height h and width a shown on the right. For any y in $[0, h]$ the cross section is given by a square. Let s denote the length of the square. Using similar triangles, we find that

$$s = \frac{a}{h}(h - y).$$

Therefore, the volume of the pyramid is given by

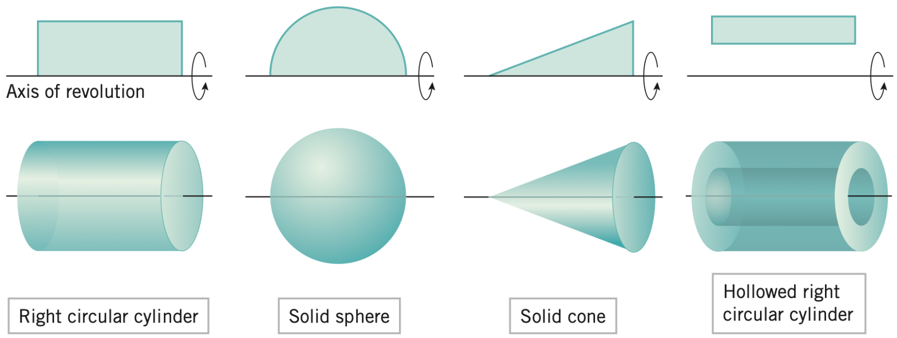
$$V = \int_0^h \frac{a^2}{h^2}(h - y)^2 dy = \frac{1}{3}a^2 h.$$



2 Solids of Revolution

A solid of revolution is formed by revolving a plane region about a line known as the axis of revolution, resulting in a solid whose cross sections are disks or washers of a certain radius.

We can determine the volume of a solid of revolution using the method of slicing. In particular, for disks the area of each cross section is given by $\pi f(x)^2$, where $f(x)$ denotes the radius of the disk. Note that $f(x)$ is the height of the original plane region measured from the axis of revolution. For washers, the area of each cross section is given by $\pi\left(f(x)^2 - g(x)^2\right)$, where $f(x)$ denotes the outer radius and $g(x)$ denotes the



(a) (b) (c) (d)

inner radius of the washer. Note that $f(x)$ and $g(x)$, respectively, denote the upper and lower bounds of the original plane region measured from the axis of revolution.

3 Exercises

Derive volume formulas for the following shapes:

a. Right circular cylinder of height h and radius r.

b. Sphere of radius r.

c. Solid cone of height h and radius r.

d. Hollowed right circular cylinder of height h, outer radius R, and inner radius r.