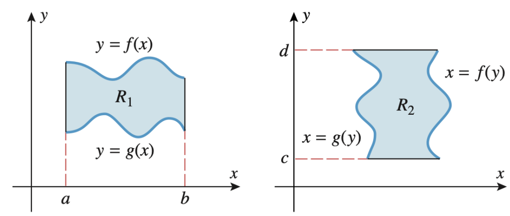
# Calculus with Analytic Geometry II

Thomas R. Cameron

January 27, 2025

## 1 Volume Review

Let $R_1$ and $R_2$ denote the plane regions shown in the figure on the right. In waht follows, we will consider situations where slicing methods or shell methods can be applied to find the volume of a solid of revolution.




Suppose that $R_1$ is revolved around the $x$-axis to form a solid of revolution. Then, slicing the solid perpendicular to the $x$-axis results in a washer with inner radius $g(x)$ and outer radius $f(x)$. Therefore, the volume of the solid can be computed as follows:

$$V = \int_a^b \pi \left(f(x)^2 - g(x)^2\right) dx.$$

Now, suppose that $R_1$ is revolved around the $y$-axis to form a solid of revolution. Then, slicing the solid perpendicular to the $y$-axis may or may not result in a washer, depending on the value of $y$. An additional issue is that $f(x)$ and $g(x)$ are not one-to-one; hence, there is no inverse function $x = f^{-1}(y)$ nor $x = g^{-1}(y)$. For this reason, the slicing methods of disks/washers is not appropriate in this case. However, the shell method may be used to compute the volume of the solid as follows:

$$V = \int_a^b 2\pi x \left(f(x) - g(x)\right) dx.$$

Similarly, if $R_2$ is revolved around the $y$-axis, then the disk/washer method can be used to determine the volume of the solid as follows:

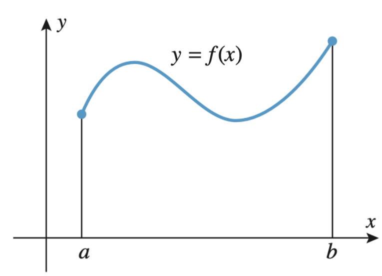
$$V = \int_c^d \pi \left(f(y)^2 - g(y)^2\right) dy.$$

Moreover, if $R_2$ is revolved around the $x$-axis, then the disk/washer method is not appropriate but the shell method can be used to determine the volume of the solid as follows:

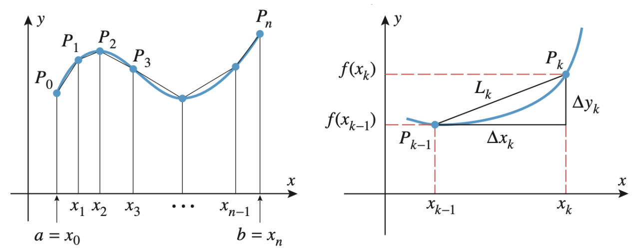
$$V = \int_c^d 2\pi x \left(f(y) - g(y)\right) dy.$$

## 2 Arc Length

Let $f(x)$ be a smooth function on $[a, b]$, i.e., $f'(x)$ is continuous on $[a, b]$. Let $L$ denote the arc length of $f(x)$ on $[a, b]$.



We can approximate $L$ by dividing the interval $[a, b]$ into $n$ subintervals $[x_{k-1}, x_k]$. Over each subinterval, we approximate the length of the curve by the line connecting the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$. We denote by $L_k$ the length of the line segment in the $k$th interval. Using Pythagorean's theorem, we find




that

$$\begin{aligned} L_k &= \sqrt{(\Delta x)^2 + (f(x_k) - f(x_{k-1}))^2} \\ &= \Delta x \sqrt{1 + \left( \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \right)^2} \\ &= \Delta x \sqrt{1 + (f'(x_k^*))^2}, \end{aligned}$$

where the last equation follows from the mean value theorem. Therefore, we can approximate the arc length as follows:

$$L \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2} \Delta x.$$

Taking the limit as $n \to \infty$ gives us

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2} \Delta x = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

Note that, if the curve is expressed as $y = f(x)$, then the arc length formula can be written as

$$L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2}.$$

Similarly, if the curve is expressed as $x = f(y)$, then the arc length formula can be written as

$$L = \int_a^b \sqrt{1 + \left( \frac{dx}{dy} \right)^2}.$$

## 3 Exercises

Find the arc length of the following curves over the given interval.

1. $f(x) = x^{3/2}$ over $[0, 2]$.
2. $y = \ln(\cos(x))$ over $[0, \pi/4]$.
3. $(y - 1)^3 = x^2$ over $[0, 8]$.