Calculus with Analytic Geometry II

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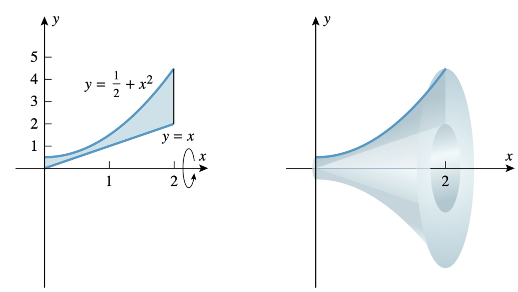
1 Solids of Revolution

A solid of revolution is formed by revolving a plane region about a line known as the axis of revolution, resulting in a solid whose cross sections are disks or washers of a certain radius. Last class, we derived formulas for the volume of several solids of revolution. Today, we consider several other examples and introduce the shell method for calculating the volume of a solid of revolution.

For example, consider the solid of revolution shown on the right. The area of each cross section of this solid is given by

$$A(x) = \pi\left(\left(\frac{1}{2}+x^2\right)^2 - (x)^2\right).$$



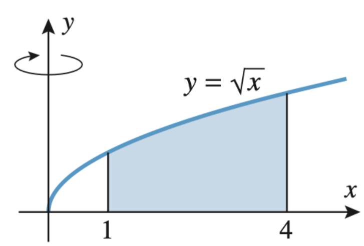
Therefore, the volume of this solid is given by

$$\begin{aligned} V &= \int_0^2 \pi\left(\left(\frac{1}{2}+x^2\right)^2 - (x)^2\right)dx \\ &= \int_0^2 \pi\left(\frac{1}{4}+x^4\right)dx \\ &= \pi\left(\frac{1}{4}x+\frac{1}{5}x^5\right)\Big|_0^2 = \pi\frac{69}{10}. \end{aligned}$$

2 Shell Method

Sometimes the method of disks and washers is not the most efficient way to find the volume of a solid of revolution.

For example, consider the solid of revolution formed by taking the plane region bounded by $y=\sqrt{x}$, $x=1$, $x=4$, and the x-axis, and revolving that region around the y-axis. Note that each cross section is a washer; however, the inner radius of the washer is $x=1$, for $0\le y\le 1$, and is $x=y^2$, for $1\le y\le 2$. Hence, to find the volume of the solid of revolution using the washer method we need two integrals.



The shell method uses cyclindrical shells to approximate the volume of a solid of revolution. A cyclindrical shell is a solid enclosed by two circular cylinders. If the outer cylinder has radius r_2 and the inner cylinder has radius r_1, then the volume of the shell is given by

$$\begin{aligned} V &= \pi\left(r_2^2-r_1^2\right)h \\ &= \pi(r_2+r_1)(r_2-r_1)h \\ &= 2\pi\left(\frac{r_1+r_2}{2}\right)(r_2-r_1)\,h. \end{aligned}$$

Now, divide the interval $[a,b]$ into n subintervals of length Δx. Then, the plane region has ben divided into n strips, which we denote by $R_1, R_2, \ldots, R_n$. When revolved around the y-axis, these strips generate tube-like solids $S_1, S_2, \ldots, S_n$ that are nested one inside the other, see Figure 1.

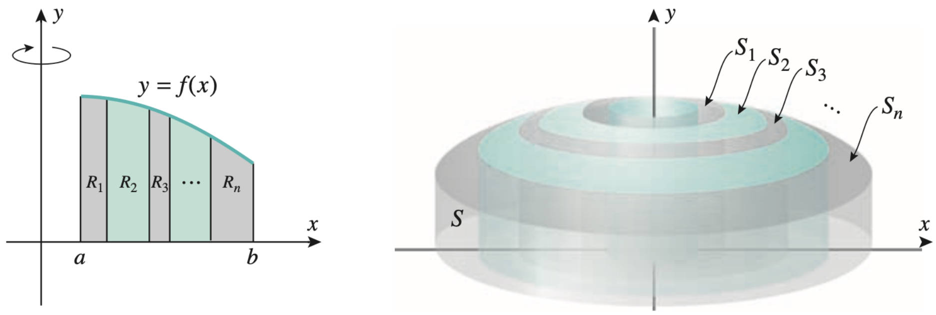


Figure 1: Shell Method, split the region into n strips and revolve around y-axis

Note that these strips have curved upper bounds. However, if the strips are thin we can approximate their areay using a rectangle. These rectangles, when revolved around the y-axis, will produce cylindrical shells whose volume closely approximate the volume of the tubes, see Figure 2.

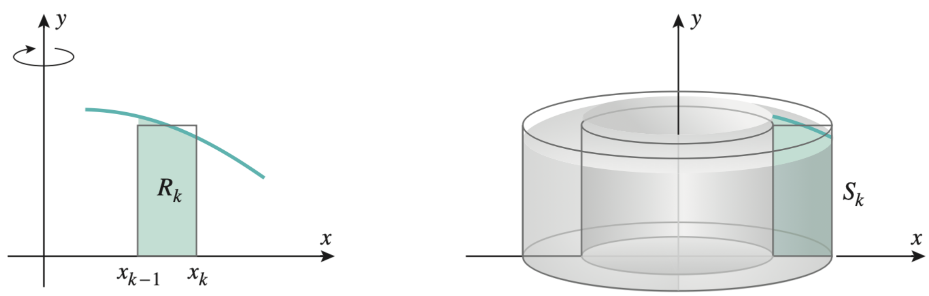


Figure 2: Shell Method, approximate the area of each strip by a rectangle and revolve around y-axis

To implement the shell method, consider the subinterval $[x_{k-1}, x_k]$ and let x_k^* denote the midpoint of this interval. If we construct a rectangle of height $f(x_k^*)$ over this interval, then revolving the rectangle around the y-axis will produce a cylindrical shell of average radius x_k^*, height $f(x_k^*)$, and thickness Δx. Hence, the volume of this shell is given by

$$V_k = 2\pi x_k^* f(x_k^*)\Delta x.$$

Therefore, we can approximate the volume of the solid of revolution as follows

$$V \approx 2\pi \sum_{i=1}^{n} x_i^* f(x_i^*)\Delta x.$$

Taking the limit as $n \to \infty$ gives us

$$V = 2\pi \lim_{n\to\infty} \sum_{i=1}^{n} x_i^* f(x_i^*)\Delta x = 2\pi \int_a^b x f(x) dx.$$

3 Exercises

Find the volume of the solid of revolution formed by taking the plane region bounded by $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x-axis, and revolving that region around the y-axis

a. Evaluate using the washer method.

b. Evaluate using the shell method.