MATH-251
Spring 2022
Exam 1 Solution
February 11

Name: ____________________

Pledge: ____________________

Each question topic and point value is recorded in the tables below. Note that this exam must be completed within the 50 minutes allotted during class. Also, you must work without any external resources, which includes no notes, calculator, nor any equivalent software. You must show an appropriate amount of work to justify your answer for each problem. If you run out of room for a given problem, you may continue your work on the back of the page. By writing your name and signing the pledge you are stating that you understand the rules outlining this exam.

Scoring Table

Question	Points	Score
1	10	
2	7	
3	7	
4	9	
5	5	
6	5	
Total:	43	

Topics Table

Question	Topic
1	1st-Order Linear
2	1st-Order Non-Linear: Separable
3	1st-Order Non-Linear: Exact
4	2nd-Order Linear Homogeneous
5	2nd-Order Linear Non-Homogeneous: Undetermined Coefficients
6	2nd-Order Linear Non-Homogeneous: Variation of Parameters

1. (a) (2 points) Explain why the following differential equation is 1st-order and linear:

$$y' + 2y = xe^{-2x}.$$

Solution: The differential equation is 1st-order since the highest order derivative is 1 and the differential equation is linear since there is no non-linear function of y and y'.

(b) (5 points) Find the general solution to the above differential equation.
Solution: The integration factor is

$$\mu(x) = e^{\int (2)dx} = e^{2x}$$

so the general solution is given by

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int \mu(x) x e^{-2x} dx \\ &= e^{-2x} \int x dx \\ &= e^{-2x} \left(\frac{x^2}{2} + C \right) \\ &= \frac{x^2}{2} e^{-2x} + Ce^{-2x}. \end{aligned}$$

(c) (3 points) Find the particular solution that satisfies the initial condition $y(0) = 2$.
Solution: Applying the initial condition gives us

$$y(0) = C = 2.$$

Hence, the particular solution is given by

$$y(x) = \frac{x^2}{2} e^{-2x} + 2e^{-2x}.$$

2. (a) (2 points) Explain why the following differential equation is separable:

$$\frac{dy}{dx} = \frac{1+y^2}{x^2}.$$

Solution: The differential equation is separable since it can be written in the form

$$-\frac{1}{x^2} + \frac{1}{1+y^2}\frac{dy}{dx} = 0,$$

i.e.,

$$-\frac{1}{x^2}dx + \frac{1}{1+y^2}dy = 0.$$

(b) (5 points) Find the general solution to the above differential equation.

Solution: The general solution can be found by integrating both sides of the above equation

$$-\int \frac{1}{x^2}dx + \int \frac{1}{1+y^2}dy = C,$$

which gives us

$$\frac{1}{x} + \arctan(y) = C.$$

We can solve the above equation for y as follows:

$$y(x) = \tan\left(C - \frac{1}{x}\right).$$

3. (a) (2 points) Explain why the following differential equation is exact:

$$(3x + 5y) + (5x - 2y)\,\frac{dy}{dx} = 0.$$

Solution: The differential equation is exact since $M(x, y) = 3x + 5y$ and $N(x, y) = 5x - 2y$ satisfy

$$M_y(x, y) = N_x(x, y).$$

(b) (5 points) Find the general solution to the above differential equation.
Solution: The general solution can be found by identifying the function $\phi(x, y)$ that satisfies

$$\phi_x(x, y) = M(x, y) \text{ and } \phi_y(x, y) = N(x, y).$$

Therefore,

$$\phi(x, y) = \int (3x + 5y)dx = \frac{3}{2}x^2 + 5xy + C(y)$$

and

$$\phi(x, y) = \int (5x - 2y)dy = 5xy - y^2 + C(x).$$

By setting $C(y) = -y^2$ and $C(x) = \frac{3}{2}x^2$ we find

$$\phi(x, y) = \frac{3}{2}x^2 + 5xy - y^2.$$

Finally, the general solution is given implicitly by $\phi(x, y) = C$, i.e.,

$$\frac{3}{2}x^2 + 5xy - y^2 = C.$$

4. Find the basis solutions to each of the following differential equations, assume that x is the independent variable.

(a) (3 points)

$$y'' + 2y' - 3y = 0.$$

Solution: The characteristic equation is given by

$$r^2 + 2r - 3 = 0 \Rightarrow (r+3)(r-1) = 0,$$

which has roots $r = -3, 1$. Hence, the basis solutions are given by

$$y_1(x) = e^{-3x} \text{ and } y_2(x) = e^{1x}.$$

(b) (3 points)

$$y'' + 2y' + y = 0.$$

Solution: The characteristic equation is given by

$$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0,$$

which has roots $r = -1, -1$. Hence, the basis solutions are given by

$$y_1(x) = e^{-x} \text{ and } y_2(x) = xe^{-x}.$$

(c) (3 points)

$$y'' + 9y = 0.$$

Solution: The characteristic equation is given by

$$r^2 + 9 = 0,$$

which has roots $r = \pm 3i$. Hence, the basis solutions are given by

$$y_1(x) = \cos(3x) \text{ and } y_2(x) = \sin(3x).$$

5. (5 points) Use the method of undetermined coefficients to find a particular solution $Y(x)$ to the following differential equation[1]:

$$y'' + 2y' - 3y = 5e^x.$$

Solution: Since the non-homogeneous part is a solution to the homogeneous equation, we have an initial guess of

$$Y(x) = Axe^x.$$

Plugging this into the differential equation gives us

$$Ae^x(x+2) + 2Ae^x(x+1) - 3Axe^x = 5e^x,$$

which can be written as

$$3Ae^x = 5e^x.$$

Hence, $A = \frac{5}{3}$ and the particular solution to the non-homogeneous equation is given by

$$Y(x) = \frac{5}{3}xe^x.$$

[1] Note that the associated homogeneous differential equation is in problem 4 (a).

6. (5 points) Use the method of variation of parameters to find a particular solution $Y(x)$ to the following differential equation[2]:

$$y'' + 9y = \frac{1}{\cos(3x)}.$$

Solution: We seek a particular solution of the form

$$Y(x) = u_1(x)\cos(3x) + u_2(x)\sin(3x).$$

In class, we derived formulas for $u_1(x)$ and $u_2(x)$, see Theorem 3.6.1 of the book. In this case, we have

$$\begin{aligned} u_1(x) &= -\frac{1}{3}\int \frac{\sin(3x)}{\cos(3x)}dx \\ &= -\frac{1}{3}\int \tan(3x)dx = \frac{1}{9}\ln\left(\cos(3x)\right) \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= \frac{1}{3}\int \frac{\cos(3x)}{\cos(3x)}dx \\ &= \frac{1}{3}\int 1dx = \frac{1}{3}x. \end{aligned}$$

Therefore, the particular solution to the non-homogeneous equation is given by

$$Y(x) = \frac{1}{9}\ln\left(\cos(3x)\right)\cos(3x) + \frac{1}{3}x\sin(3x).$$

[2]Note that the associated homogeneous differential equation is in problem 4 (c).