MATH-251 **Name:** ______________________

Spring 2022

Exam 2 Solution

March 4 **Pledge:** ______________________

Each question topic and point value is recorded in the tables below. Note that this exam must be completed within the 50 minutes allotted during class. Also, you must work without any external resources, which includes no notes, calculator, nor any equivalent software. You must show an appropriate amount of work to justify your answer for each problem. If you run out of room for a given problem, you may continue your work on the back of the page. By writing your name and signing the pledge you are stating that you understand the rules outlining this exam.

Scoring Table

Question	Points	Score
1	9	
2	8	
3	4	
4	7	
5	8	
6	7	
Total:	43	

Topics Table

Question	Topic
1	Higher-Order DEs: Undetermined Coefficients
2	Higher-Order DEs: Variation of Parameters
3	Laplace Transform: Definition
4	Laplace Transform: Solving Initial Value Problems
5	Laplace Transform: Step Functions
6	Laplace Transform: Impulse Functions

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$t^n,\ n = 0, 1, 2, \ldots$	$\dfrac{n!}{s^{n+1}}$
$\sin(at)$	$\dfrac{a}{s^2 + a^2}$
$\sinh(at)$	$\dfrac{a}{s^2 - a^2}$
$u_c(t)$	$\dfrac{e^{-cs}}{s}$
$\delta(t - c)$	e^{-cs}
$u_c(t)f(t - c)$	$e^{-cs}F(s)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

1. (a) (2 points) Explain why the following differential equation is 4th-order and linear:

$$y^{(4)} - y = e^{2t}.$$

Solution: The differential equation is 4th-order since the highest order derivative is 4 and the differential equation is linear since there is no non-linear function of y and its derivatives.

(b) (4 points) Find the basis solutions for the homogeneous equation associated with the above differential equation.

Solution: The characteristic equation is given by

$$r^4 - 1 = (r^2 - 1)(r^2 + 1) = 0,$$

which has zeros $r = \pm 1, \pm i$. Hence, the basis solutions are given by

$$y_1(t) = e^t,\ y_2(t) = e^{-t},\ y_3(t) = \cos(t),\ y_4(x) = \sin(t).$$

(c) (3 points) Use the method of undetermined coefficients to determine a particular solution $Y(t)$ to the above differential equation.

Solution: Our initial guess is of the form $Y(t) = e^{2t}$. Plugging this function into the differential equation gives us

$$16Ae^{2t} - Ae^{2t} = e^{2t},$$

which implies that $A = \frac{1}{15}$. Therefore, a particular solution to the non-homogeneous differential equation is given by

$$Y(t) = \frac{1}{15}e^{2t}.$$

2. (a) (3 points) Identify the basis solutions, $y_1(t), y_2(t), y_3(t)$, for the homogeneous equation associated with the following differential equation:

$$y''' - y' = \tan t$$

Solution: Note that the characteristic equation is given by

$$r^3 - r = r(r^2 - 1) = 0,$$

which has zeros $r = 0, \pm 1$. Hence, the basis solutions are given by

$$y_1(t) = 1, \;\; y_2(t) = e^t, \;\; y_3(t) = e^{-t}.$$

(b) (5 points) Find the functions $u_1'(t), u_2'(t), u_3'(t)$ from the method of variation of parameters, where $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + u_3(t)y_3(t)$ is a particular solution of the above differential equation.[1]

Solution: Consider the Wronskian and other related values shown below

$$W = \det\left(\begin{bmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{bmatrix}\right) = 2$$

$$W_1 = \det\left(\begin{bmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{bmatrix}\right) = -2$$

$$W_2 = \det\left(\begin{bmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{bmatrix}\right) = e^{-t}$$

$$W_3 = \det\left(\begin{bmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{bmatrix}\right) = e^t$$

Hence, the functions are given by

$$u_1'(t) = \frac{W_1}{W}\tan(t) = -\tan t$$

$$u_2'(t) = \frac{W_2}{W}\tan(t) = \frac{1}{2}e^{-t}\tan t$$

$$u_3'(t) = \frac{W_3}{W}\tan(t) = \frac{1}{2}e^{t}\tan t$$

[1] I am <u>not</u> asking you to integrate the function $u_1'(t), u_2'(t), u_3'(t)$.

3. (4 points) Use the integral definition of the Laplace transform to calculate $L\{te^t\}$.

Solution: By definition, we have

$$\begin{aligned} L\{te^t\} &= \int_0^\infty e^{-st} te^t dt \\ &= \int_0^\infty te^{t(1-s)} dt \\ &= \lim_{b\to\infty} \int_0^b te^{t(1-s)} dt \\ &= \lim_{b\to\infty} \left(\frac{t}{1-s} e^{t(1-s)} - \frac{1}{(1-s)^2} e^{t(1-s)} \right) \Bigg|_0^b \\ &= \lim_{b\to\infty} \left(\frac{b}{1-s} e^{b(1-s)} - \frac{1}{(1-s)^2} e^{b(1-s)} + \frac{1}{(1-s)^2} \right) = \frac{1}{(1-s)^2}, \end{aligned}$$

provided that $s > 1$.

4. Consider the initial value problem

$$y'' - y = \sin t,\ y(0) = 0,\ y'(0) = 0.$$

(a) (3 points) Take the Laplace transform of both sides of the differential equation shown above. Then, solve for $Y(s)$.

Solution: Taking the Laplace transform of both sides gives us

$$Y(s)(s^2 - 1) = \frac{1}{s^2+1} \Rightarrow Y(s) = \frac{1}{(s^2-1)(s^2+1)}$$

(b) (4 points) Find the inverse Laplace transform of $Y(s)$.

Solution: Using the partial fraction decomposition, we have

$$\frac{1}{(s^2-1)(s^2+1)} = \frac{As+B}{s^2-1} + \frac{Cs+D}{s^2+1},$$

i.e.,

$$1 = (As+B)(s^2+1) + (Cs+D)(s^2-1) = s^3(A+C) + s^2(B+D) + s(A-C) + (B-D).$$

Therefore, we have the following equations

$$\begin{aligned} A + C &= 0 \\ B + D &= 0 \\ A - C &= 0 \\ B - D &= 1 \end{aligned}$$

which implies that $A = 0,\ B = 1/2,\ C = 0,\ D = -1/2$. Hence, $Y(s)$ can be written as follows

$$Y(s) = \frac{1}{2}\cdot\frac{1}{s^2-1} - \frac{1}{2}\cdot\frac{1}{s^2+1},$$

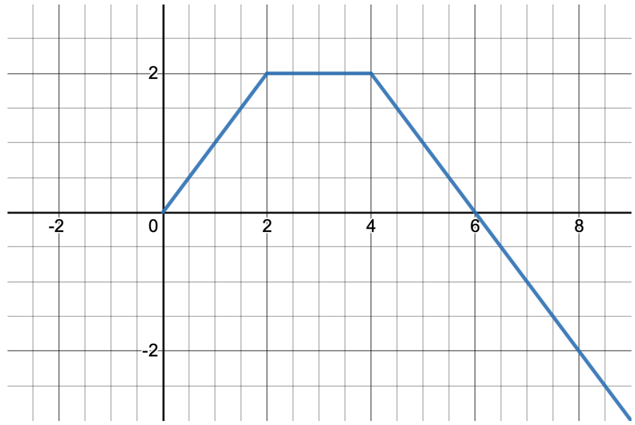
which implies that the inverse Laplace transform is given by

$$y(t) = \frac{1}{2}\sinh(t) - \frac{1}{2}\sin(t).$$

5. Consider the function

$$f(t) = \begin{cases} t & 0 \le t < 2, \\ 2 & 2 \le t < 4, \\ 6-t & t \ge 4. \end{cases}$$

(a) (2 points) Sketch a graph of $f(t)$.
Solution:



(b) (3 points) Write $f(t)$ in terms of unit step functions.
Solution:

$$\begin{aligned} f(t) &= t(1 - u_2(t)) + 2u_2(t) + (4-t)u_4(t) \\ &= t - (t-2)u_2(t) - (t-4)u_4(t) \end{aligned}$$

(c) (3 points) Find the Laplace transform of $f(t)$.
Solution: The Laplace transform is given by

$$F(s) = \frac{1}{s^2} - e^{-2s}\frac{1}{s^2} - e^{-4s}\frac{1}{s^2}.$$

6. Consider the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi),\ y(0)=0,\ y'(0)=0,$$

where $\delta(t)$ is the Dirac delta function.

(a) (3 points) Take the Laplace transform of both sides of the differential equation shown above. Then, solve for $Y(s)$.

Solution: Taking the Laplace transform of both sides gives

$$Y(s)(s^2+4) = e^{-\pi s} - e^{2\pi s} \Rightarrow Y(s) = e^{-\pi s}\frac{1}{s^2+4} + e^{-2\pi s}\frac{1}{s^2+4}.$$

(b) (4 points) Find the inverse Laplace transform of $Y(s)$.

Solution: Note that

$$Y(s) = \frac{e^{-\pi s}}{2}\frac{2}{s^2+4} - \frac{e^{-2\pi s}}{2}\frac{2}{s^2+4}.$$

Hence, the inverse Laplace transform is given by

$$\begin{aligned} y(t) &= \frac{1}{2}u_\pi(t)\sin(2(t-\pi)) - \frac{1}{2}u_{2\pi}(t)\sin(2(t-2\pi)) \\ &= \frac{1}{2}u_\pi(t)\sin(2t) - \frac{1}{2}u_{2\pi}(t)\sin(2t) \end{aligned}$$