Differential Equations

Thomas R. Cameron

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1 Daily Quiz

Find a fundamental set of solutions to the homogeneous differential equation:

$$y''' - 2y'' + 2y' = 0.$$

2 Key Topics

Today, we discuss the method of variation of parameters for finding a particular solution a higher-order non-homogeneous differential equation of the form:

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = f(t). \tag{1}$$

where $a_n \neq 0$. For further reading, see [1, Section 9.4].

2.1 Variation of Parameters

Let $\{y_1, y_2, \ldots, y_n\}$ denote a fundamental set of solutions for the complementary homogeneous differential equation to (1). We seek a particular solution to (1) of the form

$$y_p = u_1y_1 + u_2y_2 + \cdots + u_ny_n,$$

where $u_i := u_i(t)$ is differentiable for $i = 1, 2, \ldots, n$.

In addition to requiring y_p to be a particular solution, we also impose the following constraints:

$$\begin{aligned}
u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n &= 0 \\
u_1'y_1' + u_2'y_2' + \cdots + u_n'y_n' &= 0 \\
&\vdots \\
u_1'y_1^{(n-2)} + u_2'y_2^{(n-2)} + \cdots + u_n'y_n^{(n-2)} &= 0
\end{aligned}$$

Then, taking successive derivative of y_p gives us

$$\begin{aligned}
y_p' &= (u_1y_1' + u_2y_2' + \cdots + u_ny_n') + (u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n) \\
&= u_1y_1' + u_2y_2' + \cdots + u_ny_n' \\
y_p'' &= (u_1y_1'' + u_2y_2'' + \cdots + u_ny_n'') + (u_1'y_1' + u_2'y_2' + \cdots + u_n'y_n') \\
&= u_1y_1'' + u_2y_2'' + \cdots + u_ny_n'' \\
&\vdots \\
y_p^{(n-1)} &= \left(u_1y_1^{(n-1)} + u_2y_2^{(n-1)} + \cdots + u_ny_n^{(n-1)}\right) + \left(u_1'y_1^{(n-2)} + u_2'y_2^{(n-2)} + \cdots + u_n'y_n^{(n-2)}\right) \\
&= u_1y_1^{(n-1)} + u_2y_2^{(n-1)} + \cdots + u_ny_n^{(n-1)} \\
y_p^{(n)} &= u_1y_1^{(n)} + u_2y_2^{(n)} + \cdots + u_ny_n^{(n)} + u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \cdots + u_n'y_n^{(n-1)}.
\end{aligned}$$

Plugging y_p and its derivatives into (1) gives us

$$\left(u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}\right) + p_1(t)\left(u_1 y_1^{(n-1)} + \cdots + u_n y_n^{(n-1)}\right)$$
$$+ \cdots + p_{n-1}(t)\left(u_1 y_1' + \cdots + u_n y_n'\right) + p_n(t)\left(u_1 y_1 + \cdots + u_n y_n\right) = f(t).$$

Since $y_1, \ldots, y_n$ are solutions to the homogeneous differential equation, the above reduces to

$$u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = f(t).$$

Therefore, we have n equations in $u_1', \ldots, u_n'$ as follows:

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(t) \end{bmatrix}.$$

Using Cramer's rule, we can write the solution of the system as follows:

$$u_i' = \frac{f(t) W_i}{W}, \ 1 \le i \le n,$$

where W is the Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

and W_i is the resulting determinant when column i is replaced by the vector $[0,\ 0,\ \ldots,\ 1]$.

3 Exercises

Find a particular solution of

$$y''' - 2y'' + 2y' = \frac{e^t}{\cos(t) - \sin(t)}$$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.