Differential Equations

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1 Daily Quiz

2 Key Topics

Today, we introduce the Laplace transform. For further reading, see [1, Section 6.1] and [2, Section 8.1].

2.1 The Laplace Transform

Recall the improper integral
$$\int_a^\infty f(t)dt = \lim_{b\to\infty}\int_a^b f(t)dt.$$
Let $f\colon [0,\infty)\to\mathbb{R}$ be piecewise continuous. Then, the *Laplace transform* of f is defined by
$$\mathcal{L}(f) = F(s) = \int_0^\infty e^{-st}f(t)dt.$$

Example 2.1. Suppose that $f(t) = k$, where $k\in\mathbb{R}$. Then,
$$\begin{aligned}
\mathcal{L}(f) = F(s) &= \int_0^\infty ke^{-st}dt \\
&= k\int_0^\infty e^{-st}dt \\
&= -\frac{k}{s}\lim_{b\to\infty} e^{-st}\Big|_0^b \\
&= -\frac{k}{s}\lim_{b\to\infty}\left(e^{-sb}-1\right) \\
&= \frac{k}{s},
\end{aligned}$$
assuming that $s > 0$.

Example 2.2. Suppose that $f(t) = e^{t^2}$. Then,
$$\begin{aligned}
\mathcal{L}(f) &= \int_0^\infty e^{-st}e^{t^2}dt \\
&= \int_0^\infty e^{t^2-st}dt \\
&= \int_0^1 e^{t^2-st}dt + \int_1^\infty e^{t^2-st}dt \\
&\geq \int_0^1 e^{t^2-st}dt + \int_1^\infty e^{t-s}dt = \infty.
\end{aligned}$$

Note that $f(t)$ is of *exponential order* s_0 if

$$|f(t)| \leq Me^{s_0 t},\ t \geq t_0,$$

for some constants M and t_0.

Proposition 2.3. *If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then $\mathcal{L}(f)$ is defined for $s > s_0$.*

Proof. Note that

$$\begin{aligned}
|\mathcal{L}(f)| &\leq \int_0^\infty e^{-st}\,|f(t)|\,dt \\
&\leq \int_0^{t_0} e^{-st}\,|f(t)|\,dt + \int_{t_0}^\infty e^{-st}\,|f(t)|\,dt \\
&\leq \int_0^{t_0} e^{-st}\,|f(t)|\,dt + M\int_{t_0}^\infty e^{t(s_0-s)}dt,
\end{aligned}$$

which converges provided that $s > s_0$. □

We will use the Laplace transform to translate an initial value problem to an algebraic problem. Once the algebraic problem is solved, we translate the algebraic solution to the solution of the initial value problem using the inverse Laplace transform.

First, we must understand the important properties of the Laplace transform.

Proposition 2.4. *Suppose that $\mathcal{L}(f_i)$ is defined for $s > s_i$, for $1 \leq i \leq n$. Let $s_0 = \max_{1\leq i\leq n}\{s_i\}$ and let c_i be constants, for $1 \leq i \leq n$. Then,*

$$\mathcal{L}\left(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n\right) = c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2) + \cdots + c_n\mathcal{L}(f_n).$$

Proof. Note that

$$\begin{aligned}
\mathcal{L}\left(c_1 f_1 + \cdots + c_n f_n\right) &= \int_0^\infty \left(c_1 f_1 + \cdots + c_n f_n\right) e^{-st}dt \\
&= c_1\int_0^\infty e^{-st} f_1(t)dt + \cdots + c_n\int_0^\infty e^{-st} f_n(t)dt \\
&= c_1\mathcal{L}(f_1) + \cdots + c_n\mathcal{L}(f_n).
\end{aligned}$$

□

Proposition 2.5. *Suppose that $F(s) = \mathcal{L}(f)$ is defined for $s > s_0$. Then, $F(s-a)$ is the Laplace transform of $e^{at}f(t)$ for $s > s_0 + a$.*

Proof. Note that

$$\begin{aligned}
F(s-a) &= \int_0^\infty e^{-(s-a)t} f(t)dt \\
&= \int_0^\infty e^{-st}e^{at} f(t)dt \\
&= \mathcal{L}(e^{at}f(t)).
\end{aligned}$$

□

3 Exercises

Find the Laplace transform for each of the following functions

I. $f(t) = t$

II. $f(t) = e^{at}$

III. $f(t) = \cos(at)$

IV. $f(t) = \sin(at)$

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.