Differential Equations

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1 Daily Quiz

Find $\mathcal{L}(\cosh(at))$.

2 Key Topics

Today, we continue to compute the Laplace transform and we discuss the inverse Laplace transform. For further reading, see [1, Sections 8.1–8.2].

2.1 The Laplace Transform

Example 2.1.

$$\begin{aligned}
\mathcal{L}\left(\cos(at)\right) &= \int_0^\infty e^{-st}\cos(at)dt \\
&= \frac{1}{a}e^{-st}\sin(at)\Big|_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin(at)dt \\
&= \frac{1}{a}e^{-st}\sin(at)\Big|_0^\infty + \frac{s}{a}\left(-\frac{1}{a}e^{-st}\cos(at)\Big|_0^\infty - \frac{s}{a}\int_0^\infty e^{-st}\cos(at)dt\right) \\
&= \frac{1}{a}e^{-st}\sin(at)\Big|_0^\infty - \frac{s}{a^2}e^{-st}\cos(at)\Big|_0^\infty - \frac{s^2}{a^2}\int_0^\infty e^{-st}\cos(at)dt
\end{aligned}$$

Note that

$$\left(\frac{a^2+s^2}{a^2}\right)\int_0^\infty e^{-st}\cos(at)dt = \frac{1}{a}e^{-st}\sin(at)\Big|_0^\infty - \frac{s}{a^2}e^{-st}\cos(at)\Big|_0^\infty = \frac{s}{a^2}.$$

Therefore,

$$\mathcal{L}\left(\cos(at)\right) = \frac{s}{a^2+s^2}.$$

In Table 1, we give a partial table of Laplace transforms. While many more transforms could be added, this table will suffice for now.

$f(t)$	$F(s) = \mathcal{L}(f(t))$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\cos(\omega t)$	$\frac{s}{\omega^2+s^2}$
$\sin(\omega t)$	$\frac{\omega}{\omega^2+s^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-w^2}$
$\sinh(\omega t)$	$\frac{w}{s^2-w^2}$
$e^{at}f(t)$	$F(s-a)$

Table 1: Laplace Transforms

2.2 The Inverse Laplace Transform

Give $F(s)$, the inverse Laplace transform is defined by

$$f(t) = \mathcal{L}^{-1}(F(s)).$$

While there is a formula for finding the inverse Laplace transform, it requires theory of complex integration that is outside the scope of our course. We will simplify matters by using Table 1 to determine the inverse Laplace transform.

Example 2.2. Let $F(s) = \frac{1}{s^2-1}$. Setting $\omega = 1$ shows that

$$F(s) = \mathcal{L}(\sinh(\omega t)).$$

Hence, $f(t) = \mathcal{L}^{-1}(F(s)) = \sinh(t)$.

Let $F(s) = \frac{s}{s^2+9}$. Setting $\omega = 3$ shows that

$$F(s) = \mathcal{L}(\cos(\omega t)).$$

Hence, $f(t) = \mathcal{L}^{-1}(F(s)) = \cos(3t)$.

Proposition 2.3. *If $F_1, \ldots, F_n$ are Laplace transforms and $c_1, \ldots, c_n$ are constants. Then,*

$$\mathcal{L}^{-1}\left(c_1 F_1 + \cdots + c_n F_n\right) = c_1 \mathcal{L}^{-1}(F_1) + \cdots + c_n \mathcal{L}^{-1}(F_n).$$

3 Exercises

I. Complete the square and apply Proposition 2.3 to find the inverse Laplace transform of

$$F(s) = \frac{3s+8}{s^2+2s+5}.$$

II. Use partial fraction decomposition and Proposition 2.3 to find the inverse Laplace transform of

$$F(s) = \frac{3s+2}{s^2-3s+2}.$$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.