Differential Equations

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1 Daily Quiz

Find

$$\mathcal{L}^{-1}\left(\frac{2s+1}{s^2+9}\right)$$

2 Key Topics

Today, we show how the Laplace transform can be used to solve initial value problems of the form

$$ay'' + by' + cy = f(t),\ y(0) = y_0,\ y(0) = y_0'. \tag{1}$$

For further reading, see [1, Section 6.2] and [2, Section 8.3].

2.1 Laplace Transform of the Derivative

First, we must define the Laplace transform of the derivative. Recall that $y(t)$ is *exponentially bounded* if

$$|y(t)| \leq Me^{at},\ t \geq t_0$$

for some constants M, a, t_0.

Proposition 2.1. *Suppose that $y(t)$ is continuous and $y'(t)$ is piecewise continuous. If $y(t)$ is exponentially bounded, then the Laplace transform of $y'(t)$ exists for $s > a$ and*

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) = sY(s) - y(0),$$

where $Y(s)$ is the Laplace transform of $y(t)$.

Proof. Note that

$$\begin{aligned}\mathcal{L}(y') &= \int_0^\infty e^{-st}y'(t)dt \\ &= e^{-st}y(t)\Big|_0^\infty + s\int_0^\infty e^{-st}y(t)dt \\ &= s\mathcal{L}(y) - y(0).\end{aligned}$$

□

Proposition 2.2. *Suppose that $y(t)$ and $y'(t)$ are continuous and $y''(t)$ is piecewise continuous. If $y(t)$ and $y'(t)$ are exponentially bonded, then the Laplace transform of $y''(t)$ exists for $s > a$ and*

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2Y(s) - sy(0) - y'(0),$$

where $Y(s)$ is the Laplace transform of $y(t)$.

Proof. Note that

$$\begin{aligned}
\mathcal{L}(y'') &= \int_0^\infty e^{-st} y''(t) dt \\
&= e^{-st} y'(t) \Big|_0^\infty + s \int_0^\infty e^{-st} y'(t) dt \\
&= e^{-st} y'(t) \Big|_0^\infty + s \left(s\mathcal{L}(y) - y(0) \right) \\
&= s^2 \mathcal{L}(y) - sy(0) - y'(0).
\end{aligned}$$

□

2.2 Laplace Transform of the IVP

Theorem 2.3. *Suppose that the Laplace transform of y, y', y'', f exist. Then, the Laplace transform of* (1) *exists and can be written as*

$$Y(s) \left(as^2 + bs + c \right) = F(s) + (as + b) \, y(0) + ay'(0).$$

3 Exercises

Consider the initial value problem

$$y'' - 3y' + 2y = e^{3t}, \; y(0) = 1, \; y'(0) = 1.$$

I. Solve the initial value problem using the characteristic equation and the method of undetermined coefficients.

II. Solve the initial value problem using the Laplace transform.

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.