Differential Equations

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1 Daily Quiz

2 Key Topics

Today, we show how the Laplace transform can be used to solve initial value problems

$$ay'' + by' + cy = f(t),\ y(0) = y_0,\ y(0) = y_0', \tag{1}$$

where $f(t)$ is piecewise continuous (but not continuous) and exponentially bounded. For further reading, see [1, Section 8.4].

2.1 Unit Step Function

Let $c > 0$. Then the *unit step function* is defined by

$$u_c(t) = \begin{cases} 0 & 0 \leq t < c, \\ 1 & t \geq c. \end{cases}$$

We can use unit step functions to construct more interesting piecewise continuous functions.

Example 2.1.

$$u_\pi(t) - u_{2\pi}(t) = \begin{cases} 0 & 0 \leq t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$

Example 2.2.

$$2 + 3u_4(t) - 6u_7(t) + 2u_9(t) = \begin{cases} 2 & 0 \leq t < 4 \\ 5 & 4 \leq t < 7 \\ -1 & 7 \leq t < 9 \\ 1 & t \geq 9 \end{cases}$$

Example 2.3.

$$\sin(t) + u_{\pi/4}(t)\cos(t - \pi/4) = \begin{cases} \sin(t) & t < \pi/4 \\ \sin(t) + \cos(t - \pi/4) & t \geq \pi/4 \end{cases}$$

2.2 The Laplace Transform of the Unit Step Function

Theorem 2.4. *If the Laplace transform of $f(t)$ exists for $s > a \geq 0$, and if $c > 0$, then*

$$\mathcal{L}\left(u_c(t)f(t-c)\right) = e^{-cs}\mathcal{L}(f(t)),\ s > a.$$

Conversely, if $f(t)$ is the inverse Laplace transform of $F(s)$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\left(e^{-cs}F(s)\right).$$

Proof. Note that

$$\begin{aligned}\mathcal{L}\left(u_c(t)f(t-c)\right) &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\ &= \int_c^\infty e^{-st}f(t-c)dt\end{aligned}$$

Let $\sigma = t - c$, then we have

$$\begin{aligned}\mathcal{L}\left(u_c(t)f(t-c)\right) &= \int_0^\infty e^{-(\sigma+c)s}f(\sigma)d\sigma \\ &= e^{-cs}\int_0^\infty e^{-s\sigma}f(\sigma)d\sigma \\ &= e^{-cs}F(s).\end{aligned}$$

□

3 Exercises

Use the Laplace transform to solve the following initial value problem

$$y'' + y = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases},\ y(0) = 1,\ y'(0) = 1.$$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.