Differential Equations

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October 25, 2023

1 Daily Quiz

2 Key Topics

Today, we review unit step functions in the context of the Laplace transform and the inverse Laplace transform. For further reading, see [1, Section 8.4].

Let $c \geq 0$. Then, the unit step function is defined as

$$u_c(t) = \begin{cases} 0 & 0 \leq t < c \\ 1 & t \geq c \end{cases}.$$

Note that we can rewrite piecewise continuous functions as unit step functions.

Example 2.1. Let

$$f(t) = \begin{cases} 2 & 0 \leq t < \frac{\pi}{2} \\ 2 + \cos(t) & \frac{\pi}{2} \leq t < \pi \\ 0 & t \geq \pi \end{cases}.$$

Then, $f(t)$ can be rewritten as follows

$$f(t) = 2u_0(t) + \cos(t)u_{\pi/2}(t) - (2 + \cos(t))\, u_\pi(t).$$

When taking the Laplace transform of a piecewise continuous function, we make use of the following result:

$$\mathcal{L}\left(u_c(t)f(t-c)\right) = e^{-cs}F(s). \tag{1}$$

Example 2.2. Note that $f(t)$ from 2.1 can be written as follows

$$f(t) = 2u_0(t) - \sin(t - \pi/2)u_{\pi/2}(t) - (2 - \cos(t - \pi))\, u_\pi(t),$$

where $\cos(t) = -\sin(t - \pi/2)$ and $\cos(t) = -\cos(t - \pi)$ was substituted to match the form in (1). Therefore,

$$\mathcal{L}(f(t)) = \frac{2}{s} - e^{-\frac{\pi}{2}s}\frac{1}{s^2+1} - e^{-\pi s}\left(\frac{2}{s} - \frac{s}{s^2+1}\right)$$

3 Exercises

1. Find the Laplace transform of

$$f(t) = \begin{cases} \sin(t) & 0 \leq t < \frac{\pi}{2} \\ \cos(t) - 3\sin(t) & \frac{\pi}{2} \leq t < \pi \\ 3\cos(t) & t \geq \pi \end{cases}.$$

Hint: $\cos(t) = -\sin(t - \pi/2)$, $\sin(t) = \cos(t - \pi/2)$, $\cos(t) = -\cos(t - \pi)$.

2. Find the inverse Laplace transform of

$$F(s) = e^{-\frac{\pi}{2}s}\frac{s+1}{s^2+3s+2} + e^{-\pi s}\frac{s+1}{s^2+6s+10}.$$

Hint: Apply partial fraction decomposition to $\frac{s+1}{s^2+3s+2}$ and complete the square on $s^2+6s+10$.

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.