Differential Equations

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1 Daily Quiz

2 Key Topics

Today, we review the *power series*:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{1}$$

where $x_0,\ a_0,\ a_1,\ \ldots,\ a_n,\ \ldots$ are constants. For further reading, see [1, Sections 7.1].

2.1 Convergence

Given a real x, the power series (1) is said to *converge* at x if the following limit exists

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x - x_0)^n.$$

Otherwise, we say the power series *diverges* at x. Clearly, the power series (1) converges at $x = x_0$. The following theorem describes all possibilities for convergence.

Theorem 2.1. *For any power series* (1)*, exactly one of the statements is true:*

I. *The power series converges only for $x = x_0$.*

II. *The power series converges for all real x.*

III. *There is a $R > 0$ where the power series converges for all x such $|x - x_0| < R$ and diverges for all x such that $|x - x_0| > R$.*

In case III of Theorem 2.1, we say that R is the *radius of convergence* for the power series. In case I, $R = 0$ and in case II, $R = \infty$.

The following theorem gives a useful test for convergence, known as the ratio test.

Theorem 2.2. *Suppose there is an N such that $a_n \neq 0$ for all $n \geq N$. If*

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

then the power series (1) *has radius of convergence $R = 1/L$, where $R = 0$ if $L = \infty$ and $R = \infty$ if $L = 0$.*

Note that if $|x - x_0| < R = 1/L$, then we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x - x_0| < 1.$$

2.2 Examples

Example 2.3. Consider the power series

$$\sum_{n=0}^{\infty} n! x^n .$$

Applying the ratio test gives

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty .$$

Hence, the given power series has radius of convergence $R = 0$, i.e., the power series only converges for $x = 0$.

Example 2.4. Consider the power series

$$\sum_{n=0}^{\infty} 2^n n^2 (x-1)^n .$$

Applying the ratio test gives

$$\begin{aligned}\lim_{n\to\infty} \frac{2^{n+1}(n+1)^2}{2^n n^2} &= 2 \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} \\ &= 2 \lim_{n\to\infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = 2.\end{aligned}$$

Therefore, the given power series has radius of convergence $R = 1/2$, i.e., the power series only converges for $x \in (1/2, 3/2)$.

Example 2.5. Consider the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

Applying the ratio test gives

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{(-1)^{n+1}/(n+1)!}{(-1)^n/n!} \right| &= \lim_{n\to\infty} \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{1}{n+1} = 0.\end{aligned}$$

Hence, the given power series has radius of convergence $R = \infty$, i.e., the power series converges for all x.

3 Exercises

Find the radius of convergence for all power (Taylor) series below.

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
3. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
4. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.