Differential Equations

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1 Daily Quiz

Find the Taylor series of

$$f(x) = \frac{1}{1-x}.$$

2 Key Topics

Today, we continue our review of *power series*:

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n, \tag{1}$$

where x_0, $a_0,\ a_1,\ \ldots,\ a_n,\ \ldots$ are constants. In particular, we will discuss differentiation of power series and the shifting of the index. For further reading, see [1, Sections 7.1].

2.1 Differentiation

Theorem 2.1. *Suppose the power series* (1) *has radius of convergence $R > 0$. Then, all derivatives of the power series exist in the open interval centered at x_0 with radius R. Furthermore, the derivatives can be attained with term by term differentiation.*

Example 2.2. Let $f(x) = \sin(x)$. Then,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ f'(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos(x). \end{aligned}$$

Theorem 2.3. *Suppose that the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

has a positive radius of convergence, i.e., $R > 0$. Then,

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

i.e., $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is the Taylor series of f at x_0.

Proof. Note that

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &= a_0 + \sum_{n=1}^{\infty} a_n (x - x_0)^n , \end{aligned}$$

so $f(x_0) = a_0$. Next, note that

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \\ &= 0 + a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} (x - x_0)^n , \end{aligned}$$

so $a_1 = f'(x_0)$. Next, note that

$$\begin{aligned} f''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \\ &= 0 + 0 + 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n , \end{aligned}$$

so $a_2 = f''(x_0)/2$. In general, we see that the coefficients of the power series are uniquely defined by the formula

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

□

Theorem 2.3 implies that if a power series has a positive radius of convergence, then the series is a unique.

2.2 Shifting Index

There several occasions during the proof of Theorem 2.3 where we shifted the index of the power series. For example,

$$\sum_{n=2}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=1}^{\infty} (n+1) a_{n+1} (x - x_0)^n$$

is obtained from the shift $n = n + 1$. Similarly,

$$\sum_{n=3}^{\infty} n(n-1) a_n (x - x_0)^{n-2} = \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n$$

is obtained from the shift $n = n + 2$.

Shifting the series index is very useful when combining multiple series. For example,

$$\begin{aligned} \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} + a_n \right] x^n. \end{aligned}$$

3 Exercises

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a positive radius of convergence. Identify the coefficients $a_0, a_1, a_2, \ldots$ that satisfy $f'(x) = f(x)$.

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.