Differential Equations

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1 Daily Quiz

Find the formula for u_1' and u_2' in variation of parameters applied to

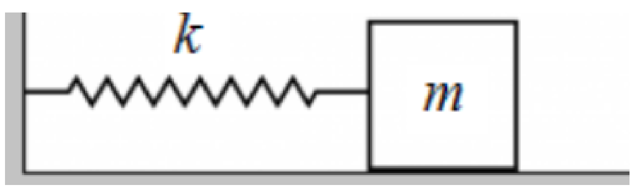
$$y'' + 3y' + 2y = \frac{1}{1+e^t}.$$

2 Key Topics

Today, we revisit applications of second-order differential equations to the study of harmonics. For further reading, see [2, Section 6.2] or [1, Section 4.3].

2.1 Mass Spring System

Consider the mass spring system shown in Figure 2.1, winch shows a mass m connected to a wall via a spring with spring constant k.



Assuming the mass is sliding on a surface with a friction coefficient b, the following differential equation describes the displacement from equilibrium of the mass:

$$my'' + by' + ky = f(t),$$

where $f(t)$ is an external forcing function.

Example 2.1. Consider the differential equation

$$y'' + 6y' + 5y = \sin(2t).$$

A fundamental set of solutions to the homogeneous equation is given by

$$y_1 = e^{-t},\ y_2 = e^{-5t}.$$

A particular solution will be of the form

$$y_p = A\cos(2t) + B\sin(2t).$$

Plugging y_p into the differential equation gives

$$\begin{aligned} y_p'' + 6y_p' + 5y_p &= (-4A\cos(2t) - 4B\sin(2t)) + 6\left(-2A\sin(2t) + 2B\cos(2t)\right) + 5\left(A\cos(2t) + B\sin(2t)\right) \\ &= \cos(2t)\left(A + 12B\right) + \sin(2t)\left(B - 12A\right). \end{aligned}$$

Therefore, A and B must satisfy

$$\begin{aligned} A + 12B &= 0 \\ -12A + B &= 1 \end{aligned}$$

which implies that $B = 1/145$ and $A = -12/145$. Therefore, the general solution is given by

$$y = -\frac{12}{145}\cos(2t) + \frac{1}{145}\sin(2t) + c_1 e^{-t} + c_2 e^{-5t}$$

2.2 Amplitude, Frequency, and Phase Shift

In Example 2.1, note that the general solution approaches the particular solution as $t \to \infty$. For this reason, we would like to rewrite the particular solution in the following form

$$y_p(t) = R\cos(\omega t - \phi),$$

where R denotes the amplitude, ω the frequency, and ϕ the phase shift of the particular solution.

To this end, note that

$$R\cos(\omega t - \phi) = R\cos(\phi)\cos(\omega t) + R\sin(\phi)\sin(\omega t).$$

Hence, by comparing the above form to

$$A\cos(\omega t) + B\sin(\omega t),$$

we see that

$$A^2 + B^2 = R^2$$

and

$$A = R\cos(\phi),\ \ B = R\sin(\phi).$$

Example 2.2. We will rewrite the particular solution from Example 2.1 in the following form

$$y_p(t) = R\cos(\omega t - \phi).$$

To this end, note that

$$R^2 = \left(-\frac{12}{145}\right)^2 + \left(\frac{1}{145}\right)^2 = \frac{1}{145},$$

so $R = 145^{-1/2}$. Furthermore,

$$\tan(\phi) = -\frac{1}{12},$$

where ϕ must be chosen to be in the second quadrant since cosine is negative and sine is positive. so $\phi = \arctan(-1/12) + \pi \approx 3.058$. Therefore, our particular solution is of the form

$$y_p(t) = \frac{1}{\sqrt{145}}\cos\left(2t - \phi\right) \approx \frac{1}{\sqrt{145}}\cos\left(2t - 3.058\right).$$

3 Exercises

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.