Differential Equations

Thomas R. Cameron

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1 Daily Quiz

Find the formula for u_1' and u_2' in variation of parameters applied to

$$y'' + 3y' + 2y = \frac{1}{1+e^t}.$$

2 Key Topics

Today, we begin our discussion of linear high-order differential equations:

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = f(t). \tag{1}$$

For further reading, see [1, Section 9.1].

2.1 Existence and Uniqueness

Theorem 2.1. *Suppose that $f(t)$ and $p_i(t)$, $1 \leq i \leq n$ are continuous on the interval (a,b). Then, the differential equation in (1) has a unique solution on (a,b) that satisfies the initial values*

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)},$$

where $t_0 \in (a,b)$, and $y_0, y_0', \cdots, y_0^{(n-1)} \in \mathbb{R}$.

Theorem 2.2. *Let $y_1, y_2, \ldots, y_n$ be solutions to the complementary homogeneous differential equation, i.e., where $f(t) = 0$. Then, for any constants $c_1, c_2, \ldots, c_n$,*

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is also a solution of the homogeneous differential equation.

Proof. Note that the kth derivative of y can be written as

$$\begin{aligned} y^{(k)} &= c_1y_1^{(k)} + \cdots + c_ny_n^{(k)} \\ &= \sum_{i=1}^n c_iy_i^{(k)}. \end{aligned}$$

Therefore, plugging y into the differential equation in (1) gives

$$\begin{aligned} &\sum_{i=1}^n c_iy_i^{(n)} + p_1(t)\sum_{i=1}^n c_iy_i^{(n-1)} + \cdots + p_{n-1}(t)\sum_{i=1}^n c_iy_i' + p_n(t)\sum_{i=1}^n c_iy_i \\ &= c_1\left(y_1^{(n)} + p_1(t)y_1^{(n-1)} + \cdots + p_{n-1}(t)y_1' + p_n(t)y_1\right) \\ &+ \cdots \\ &+ c_n\left(y_n^{(n)} + p_1(t)y_n^{(n-1)} + \cdots + p_{n-1}(t)y_n' + p_n(t)y_n\right) \\ &= 0 + 0 + \cdots + 0 = 0. \end{aligned}$$

□

We say that $\{y_1, \ldots, y_n\}$ forms a fundamental set of solutions to the complementary homogeneous differential equation if every solution to the homogeneous differential equation can be written in the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n, \tag{2}$$

for some constants $c_1, c_2, \ldots, c_n$.

Let $y(t)$ be the unique solution guaranteed by Theorem 2.1. Then, $y(t)$ can be written in the form of (2) if and only if the following system of equations has a unique solution:

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned} \tag{3}$$

The system of equations in (3) has a unique solution if and only if

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix} \neq 0. \tag{4}$$

Therefore, the solutions y_i, $1 \leq i \leq n$ to the complementary homogeneous differential equation form a fundamental set if and only if (4) holds for all $t_0 \in (a, b)$.

We define the Wronskian of $\{y_1, \ldots, y_n\}$ as follows

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}. \tag{5}$$

It is important to note that the Wronskian can also be written as

$$W(y_1, \ldots, y_n) = Ce^{-\int p_1(t)dt}, \tag{6}$$

for some constant C. Hence, the Wornskian is either always zero or never zero.

We summarize these results in the following theorem.

Theorem 2.3. *Suppose that $p_i(t)$, $1 \leq i \leq n$, are continuous on (a, b). Then, the homogeneous solutions y_i, $1 \leq i \leq n$, form a fundamental set on (a, b) if and only if the Wronskian $W(y_1, \ldots, y_n) \neq 0$.*

3 Exercises

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.