Differential Equations

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1 Daily Quiz

Show that $y = e^t$ is a solution of the homogeneous differential equation:

$$y''' + 3y'' - 2y' - 2y = 0.$$

2 Key Topics

Today, we form the fundamental set of solutions to the homogeneous constant coefficient differential equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{1}$$

where $a_n \neq 0$. For further reading, see [1, Section 9.2].

2.1 Characteristic Equation

We begin by seeking solutions to (1) of the form $y = e^{rt}$. Plugging into the differential equation gives

$$\begin{aligned} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y &= a_n r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + \cdots + a_1 r e^{rt} + a_0 e^{rt} \\ &= e^{rt} \left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 \right). \end{aligned}$$

Hence $y = e^{rt}$ is a solution to (1) if and only if

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{2}$$

We reference (2) as the characteristic equation of (1).

Suppose that $r_1, \ldots, r_n$ are distinct roots of the characteristic equation, then the following functions

$$y_1 = e^{r_1 t},\ y_2 = e^{r_2 t},\ \ldots,\ y_k = e^{r_n t} \tag{3}$$

form a fundamental set of solutions.

All complex roots of the characteristic equation must come in complex conjugate pairs

$$r = w \pm zi.$$

The corresponding solutions can be written in the form

$$y_1 = e^{wt} \cos(zt),\ y_2 = e^{wt} \sin(zt). \tag{4}$$

Suppose that r is a repeated root of the characteristic equation, with multiplicity m. Then, the corresponding solutions can be written in the form

$$y_1 = e^{rt},\ y_2 = te^{rt},\ \ldots,\ y_m = t^{m-1} e^{rt}. \tag{5}$$

2.2 The Wronskian

To establish that we have a fundamental set of solutions, we must prove that the Wronskian is non-zero. We will demonstrate that this is the case through several examples.

Example 2.1. Consider the homogeneous differential equation

$$y''' - 2y'' - 5y' + 6y = 0.$$

The characteristic equation can be written as follows

$$r^3 - 2r^2 - 5r + 6 = (r-1)(r+2)(r-3) = 0.$$

Hence, we have the following fundamental set of solutions:

$$y_1 = e^{-2t}, y_2 = e^t, y_3 = e^{3t}.$$

We can verify that $\{y_1, y_2, y_3\}$ is a fundamental set by showing that the Wronskian is non-zero. Since the Wronskian is either always zero or never zero, we will evaluate the Wronskian at $t = 0$:

$$\begin{aligned} W(0) &= \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 1 & 9 \end{vmatrix} \qquad (2r_1 + r_2, \quad -4r_1 + r_3) \\ &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & -3 & 5 \end{vmatrix} \qquad (r_2 + r_3) \\ &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 10 \end{vmatrix} = 30. \end{aligned}$$

Note that we are using Gaussian elimination to reduce the 3×3 matrix to an upper triangular form at which point the determinant is equal to the product of the diagonal entries.

Example 2.2. Consider the homogeneous differential equation

$$y^{(5)} - y^{(4)} - y' + y = 0.$$

The characteristic equation can be written as follows

$$r^5 - r^4 - r + 1 = (r+1)(r-1)^2(r^2+1) = 0.$$

Hence, we have the following set of fundamental solutions:

$$y_1 = e^{-t},\ y_2 = e^t,\ t_3 = te^t,\ y_4 = \cos(t),\ y_5 = \sin(t).$$

Again, we evaluate the Wronskian t $t = 0$:

$$W(0) = \begin{vmatrix} 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & -1 & 0 \\ -1 & 1 & 3 & 0 & -1 \\ 1 & 1 & 4 & 1 & 0 \end{vmatrix} \qquad (r_1 + r_2, \; -r_1 + r_3, \; r_1 + r_4, \; -r_1 + r_5)$$

$$= \begin{vmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 2 & 3 & 1 & -1 \\ 0 & 0 & 4 & 0 & 0 \end{vmatrix} \qquad (-r_2 + r_4)$$

$$= \begin{vmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 4 & 0 & 0 \end{vmatrix} \qquad (-r_3 + r_4, \; -2r_3 + r_5)$$

$$= \begin{vmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 & 0 \end{vmatrix} \qquad (-2r_4 + r_5)$$

$$= \begin{vmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = 32.$$

3 Exercises

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.