Differential Equations

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1 Daily Quiz

2 Key Topics

Today, we give examples on how power series can be used to solve differential equations. For further reading, see [1, Sections 7.2].

2.1 Examples

Example 2.1. Consider the differential equation

$$y' - y = 0,$$

and assume that the solution can be written as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

with a positive radius of convergence. Then, all derivatives of y exist and can be attained with term-by-term differentiation. In particular,

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Plugging the power series of y and y' into the differential equation gives

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - a_n\right] x^n. \end{aligned}$$

Therefore, $a_{n+1} = \frac{a_n}{n+1}$, for $n \geq 0$. Let a_0 be an arbitrary constant, then

$$\begin{aligned} a_1 &= \frac{a_0}{1} \\ a_2 &= \frac{a_1}{2} = \frac{a_0}{2 \cdot 1} \\ &\vdots \\ a_n &= \frac{a_0}{n!} \end{aligned}$$

So, our power series solution is of the form

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n = a_0 e^x.$$

Example 2.2. Consider the differential equation

$$y'' + y = 0,$$

and assume that the solution can be written as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

with a positive radius of convergence. Then, all derivatives of y exist and can be attained with term-by-term differentiation. In particular,

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Plugging the power series of y and y'' into the differential equation gives

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n. \end{aligned}$$

Therefore, $a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$, for $n \geq 0$. Let a_0 and a_1 be arbitrary constants, then

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 1} \qquad & a_3 &= -\frac{a_1}{3 \cdot 2} \\ a_4 &= \frac{a_0}{4!} \qquad & q_5 &= \frac{a_1}{5!} \\ &\vdots \\ a_{2n} &= (-1)^n \frac{a_0}{(2n)!} \qquad & a_{2n+1} &= (-1)^n \frac{a_1}{(2n+1)!} \end{aligned}$$

So, our power series solution is of the form

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = a_0 \cos(x) + a_1 \sin(x).$$

Example 2.3. Consider the differential equation

$$y'' - xy = 0,$$

and assume that the solution can be written as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

with a positive radius of convergence. Then, all derivatives of y exist and can be attained with term-by-term differentiation. In particular,

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Also, note that

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Plugging the power series of xy and y'' into the differential equation gives

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n \\ &= 2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1}\right]x^n \end{aligned}$$

Therefore, $a_2 = 0$ and $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$ It follows that

$$\begin{aligned} a_{3n} &= \frac{a_0}{(2\cdot 3)\cdot(5\cdot 6)\cdots((3n-1)\cdot(3n))} \\ a_{3n+1} &= \frac{a_1}{(3\cdot 4)\cdot(6\cdot 7)\cdots((3n)\cdot(3n+1))} \\ a_{3n+2} &= 0, \end{aligned}$$

for $n \geq 1$, where a_0 and a_1 are arbitrary constants.

3 Exercises

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.