Differential Equations

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1 Daily Quiz

2 Key Topics

Today, we introduce boundary value problems. For further reading, see [1, Sections 11.1].

2.1 Boundary Value Problems

A boundary value problem is a one-parameter family of differential equations associated with boundary conditions. For example, consider the boundary value problem

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(L) = 0, \tag{1}$$

where λ is a parameter and $L > 0$.

Note that $y = 0$ is always a solution of (1). However, we are interested in non-trivial solutions. We say that λ is an *eigenvalue* if (1) has a non-trivial solution, any such non-trivial solution is a corresponding *eigenfunction*.

We will consider cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ to determine the eigenvalues and corresponding eigenfunctions of (1).

- If $\lambda < 0$, then the characteristic equation $r^2 + \lambda = 0$ has two distinct real roots $r = \pm\sqrt{-\lambda} = \pm u$. Hence, the general solution to the differential equation is given by

  $$y = c_1 e^{ux} + c_2 e^{-ux}.$$

  Applying the boundary conditions gives

  $$\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y(L) &= c_1 e^{uL} + c_2 e^{-uL} = 0, \end{aligned}$$

  which implies that $c_1 = 0$ and $c_2 = 0$. Therefore, if $\lambda < 0$, then only the trivial solution exists.

- If $\lambda = 0$, then the characteristic equation has two repeated roots $r = 0, 0$. Hence, the general solution to the differential equation is given by

  $$y = c_1 + c_2 x.$$

  Applying the boundary conditions gives

  $$\begin{aligned} y(0) &= c_1 = 0 \\ y(L) &= c_1 + c_2 L = 0, \end{aligned}$$

  which implies that $c_1 = 0$ and $c_2 = 0$. Therefore, if $\lambda = 0$, then only the trivial solution exists.

- If $\lambda > 0$, then the characteristic equation has two imaginary roots $r = \pm i\sqrt{\lambda}$. Hence, the general solution to the differential equation is given by

$$y = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

  Applying the boundary conditions gives

$$\begin{aligned} y(0) &= c_1 = 0 \\ y(L) &= c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L) = 0, \end{aligned}$$

  which implies that $c_1 = 0$ and $c_2 \neq 0$ whenever $\sqrt{\lambda}L = n\pi$, where n is a positive integer.

Therefore, the eigenvalues of (1) are given by

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

and corresponding eigenfunctions are given by

$$y_n = \sin\left(\frac{n\pi}{L}x\right)$$

3 Exercises

Find the eigenvalues and corresponding eigenfunctions of

$$y'' + \lambda y = 0,\ y'(0) = 0,\ y'(L) = 0. \tag{2}$$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.