# Differential Equations

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## 1 Daily Quiz

Show that $\lambda = 0$ is an eigenvalue of

$$y'' + \lambda y = 0,\ y'(0) = 0,\ y'(L) = 0$$

## 2 Key Topics

Today, we continue solving eigenvalue problems associated with boundary value problems. In addition, we introduce the notion of orthogonal functions. For further reading, see [1, Sections 11.1].

### 2.1 Boundary Value Problems

Last time, we ended with the following boundary value problem Find the eigenvalues and corresponding eigenfunctions of

$$y'' + \lambda y = 0,\ y'(0) = 0,\ y'(L) = 0. \tag{1}$$

We consider cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ to determine the eigenvalues and corresponding eigenfunctions of (1).

- If $\lambda < 0$, then the general solution is given by

  $$y = c_1 e^{ux} + c_2 e^{-ux},$$

  where $u = \sqrt{-\lambda} > 0$. Applying the boundary conditions gives

  $$\begin{aligned} y'(0) &= c_1 u - c_2 u = 0 \\ y'(L) &= c_1 u e^{uL} - c_2 u e^{-uL} = 0, \end{aligned}$$

  which implies that $c_1 = 0$ and $c_2 = 0$. Therefore, if $\lambda < 0$, then only the trivial solution exists.

- If $\lambda = 0$, then the general solution is given by

  $$y = c_1 + c_2 x.$$

  Applying the boundary conditions gives

  $$\begin{aligned} y'(0) &= c_2 = 0 \\ y'(L) &= c_2 = 0, \end{aligned}$$

  which implies that $c_1$ can be anything. Therefore, $\lambda = 0$ is an eigenvalue of (1) with corresponding eigenfunction $y = 1$.

- If $\lambda > 0$, then the general solution is given by

$$y = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

Applying the boundary conditions gives

$$\begin{aligned} y'(0) &= \sqrt{\lambda}c_2 = 0 \\ y'(L) &= -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}L) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}L) = 0, \end{aligned}$$

which implies that $c_2 = 0$ and $c_1 \neq 0$ whenever $\sqrt{\lambda}L = n\pi$, where $n$ is a positive integer. Therefore,

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \ n = 1, 2, 3, \ldots,$$

are eigenvalues of (1) with corresponding eigenfunctions

$$y_n = \cos\left(\frac{n\pi}{L}x\right), \ n = 1, 2, 3, \ldots.$$

We summarize the eigenvalues and eigenfunctions for the two boundary value problems we've investigated thus far.

**Theorem 2.1.** *The eigenvalues and eigenfunctions for the boundary value problem*

$$y'' + \lambda y = 0, \ y(0) = 0, \ y(L) = 0$$

*are given by*

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \ y_n = \sin\left(\frac{n\pi}{L}x\right), \ n = 1, 2, 3, \ldots.$$

**Theorem 2.2.** *The eigenvalues and eigenfunctions for the boundary value problem*

$$y'' + \lambda y = 0, \ y'(0) = 0, \ y'(L) = 0$$

*are given by*

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \ y_n = \cos\left(\frac{n\pi}{L}x\right), \ n = 0, 1, 2, 3, \ldots.$$

## 2.2 Orthogonal Functions

Two integrable functions $f$ and $g$ are said to be *orthogonal* on $[a, b]$ if

$$\int_a^b f(x)g(x)dx = 0.$$

Moreover, a collection of integrable functions $f_1, \ f_2, \ f_3, \ \ldots$ is said to be mutually orthogonal on $[a, b]$ if

$$\int_a^b f_i(x)f_j(x)dx = 0$$

for all $i \neq j$.

*Example* 2.3. Let $f(x) = x$ and $g(x) = x^2$. Then, $f$ and $g$ are orthogonal on $[-1, 1]$. Indeed, note that

$$\begin{aligned} \int_{-1}^1 f(x)g(x)dx &= \int_{-1}^1 x^3 dx \\ &= \frac{1}{4}x^4\bigg|_{-1}^1 \\ &= \frac{1}{4}(1 - 1) = 0. \end{aligned}$$

## 3 Exercises

Show that the eigenfunctions in Theorem 2.1 and Theorem 2.2 are mutually orthogonal.

## References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.