Differential Equations

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1 Daily Quiz

Show that $\lambda = 0$ is an eigenvalue of

$$y'' + \lambda y = 0,\ y(-L) = y(L),\ y'(-L) = y'(L).$$

2 Key Topics

Today, we introduce the Fourier series:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right), \tag{1}$$

where $L > 0$. For further reading, see [1, Sections 11.2].

2.1 Orthogonality of Sine and Cosine Functions

The functions $\{\cos\left(\frac{n\pi}{L}x\right),\ \sin\left(\frac{n\pi}{L}x\right)\}_{n=1}^{\infty}$ are mutually orthogonal on $[-L, L]$. Indeed, we have

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = 0$$

$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

2.2 Euler-Fourier Formulas

Suppose that the Fourier series in (1) converges for every $x \in [-L, L]$. Further, let $f(x)$ denote the value of the series for each $x \in [-L, L]$, i.e.,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right). \tag{2}$$

Now, multiply both sides of (2) by $\cos\left(\frac{m\pi}{L}x\right)$, where $m \geq 0$ is a fixed integer. Then, integrate with respect to x from $-L$ to L:

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx &= \frac{1}{2}a_0\int_{-L}^{L}\cos\left(\frac{m\pi}{L}x\right)dx \\ &\quad + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx + b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx\right)\end{aligned}$$

If $m = 0$, then we have

$$\int_{-L}^{L} f(x)dx = a_0 L,$$

hence, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$. If $m \geq 1$, then we have

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx = a_m L,$$

hence, $a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx$.

Similarly, multiply both sides of (2) by $\sin\left(\frac{m\pi}{L}x\right)$, where $m \geq 0$ is a fixed integer. Then, integrate with respect to x from $-L$ to L:

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx = \frac{1}{2}a_0\int_{-L}^{L}\sin\left(\frac{m\pi}{L}x\right)dx$$
$$+\sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx + b_n\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx\right)$$

If $m = 0$, then we have $0 = 0$. If $m \geq 1$, then we have

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx = b_m L,$$

hence, $b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx$.

3 Exercises

I. Prove the orthogonality formulas given in Section 2.1.

II. Find the Fourier series of

$$f(x) = \begin{cases} -x & -2 \leq x < 0 \\ x & 0 \leq x < 2 \end{cases}$$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.