Differential Equations

Thomas R. Cameron

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1 Daily Quiz

2 Key Topics

Today, we review some basic examples of Fourier series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right), \tag{1}$$

where $L > 0$. For further reading, see [1, Sections 11.2].

Recall that the coefficients of the Fourier series in (1) can be computed as follows:

$$\begin{aligned} a_m &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx,\ m \geq 0 \\ b_m &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx,\ m \geq 1 \end{aligned}$$

2.1 Examples

Example 2.1. Last time we began the computation of the Fourier series of

$$f(x) = \begin{cases} -x & -2 \leq x < 0 \\ x & 0 \leq x < 2 \end{cases}$$

Note that

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)dx = 2, \\ a_m &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{m\pi}{2}x\right)dx,\ m \geq 1, \\ b_m &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{m\pi}{2}x\right)dx,\ m \geq 1. \end{aligned}$$

Since $\sin(x)$ is an odd function and $f(x)$ is an even function, it follows that $b_m = 0$ for all $m \geq 1$. For $m \geq 1$, we have

$$\begin{aligned} a_m &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{m\pi}{2}x\right)dx,\ m \geq 0 \\ &= \frac{1}{2}\left(-\int_{-2}^{0} x\cos\left(\frac{m\pi}{2}x\right)dx + \int_{0}^{2} x\cos\left(\frac{m\pi}{2}dx\right)\right) \end{aligned}$$

Using integration by parts, we find that

$$\begin{aligned}\int x\cos\left(\frac{m\pi}{2}x\right)dx &= \frac{2x}{m\pi}\sin\left(\frac{m\pi}{2}x\right) - \frac{2}{m\pi}\int \sin\left(\frac{m\pi}{2}x\right)dx \\ &= \frac{2x}{m\pi}\sin\left(\frac{m\pi}{2}x\right) + \frac{4}{m^2\pi^2}\cos\left(\frac{m\pi}{2}x\right)\end{aligned}$$

Therefore,

$$\begin{aligned}a_m &= \frac{1}{2}\left(-\int_{-2}^{0} x\cos\left(\frac{m\pi}{2}x\right)dx + \int_0^2 x\cos\left(\frac{m\pi}{2}dx\right)\right) \\ &= \frac{1}{2}\left(-\left(\frac{4}{m^2\pi^2} - \frac{4}{m^2\pi^2}(-1)^m\right) + \left(\frac{4}{m^2\pi^2}(-1)^m - \frac{4}{m^2\pi^2}\right)\right) \\ &= \begin{cases} 0 & m \text{ is even} \\ -\frac{8}{m^2\pi^2} & m \text{ is odd}\end{cases}\end{aligned}$$

So, the Fourier series of $f(x)$ is given by

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

3 Exercises

Find the Fourier series of $f(x) = x$ on $[-2, 2]$.

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.