Differential Equations

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1 Daily Quiz

2 Key Topics

Today, we discuss when a function $f(x)$ can be represented by a Fourier series. To this end, we denote the Fourier series of f on $[-L, L]$ by

$$F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right), \tag{1}$$

where $L > 0$. For further reading, see [1, Sections 11.2].

Recall that the coefficients of the Fourier series in (1) can be computed as follows:

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx,\ m \geq 0$$
$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi}{L}x\right)dx,\ m \geq 1$$

2.1 Examples

Example 2.1. Recall that the Fourier series of

$$f(x) = \begin{cases} -x & -2 \leq x < 0 \\ x & 0 \leq x < 2 \end{cases} \tag{2}$$

on the interval $[-2, 2]$ is given by

$$F(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

From Figure 1, it is clear that the Fourier series converges and $F(x) = f(x)$ for all $x \in [-2, 2]$.

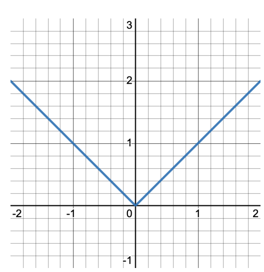

Figure 1: Fourier Series of (2) on the interval $[-2, 2]$

Example 2.2. Recall that the Fourier series of $f(x) = x$ on the interval $[-2, 2]$ is given by

$$F(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2} x\right)$$

From Figure 2, it is clear that the Fourier series converges and $F(x) = f(x)$ for all $x \in (-2, 2)$. However, at the endpoints, i.e., when $x = -2$ or $x = 2$, $F(x) \neq f(x)$. In fact, if we let $x = -2$ or $x = 2$, then the Fourier

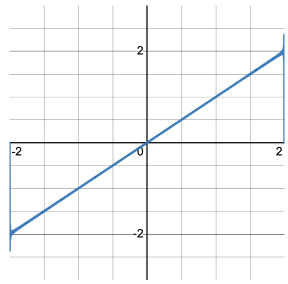

Figure 2: Fourier Series of $f(x) = x$ on the interval $[-2, 2]$

series becomes

$$F(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} 0 = 0.$$

2.2 Convergence of Fourier Series

The following theorem summarizes the convergence of the Fourier series of a piecewise smooth function. Note that

$$f(c^-) = \lim_{x \to c^-} f(x)$$

and

$$f(c^+) = \lim_{x \to c^+} f(x)$$

Theorem 2.3. *Suppose that f and f' are piecewise continuous on $[-L, L]$. Then, the Fourier series of f on $[-L, L]$ converges. Moreover,*

- *if $-L < c < L$ and f is continuous at c, then $F(c) = f(c)$,*
- *if $-L < c < L$ and f is discontinuous at c, then $F(c) = \frac{f(c^-)+f(c^+)}{2}$,*
- *$F(-L) = F(L) = \frac{f(-L^+)+f(L^-)}{2}$.*

3 Exercises

Find the Fourier series of

$$f(x) = \begin{cases} -x & -2 \leq x < 0 \\ \frac{1}{2} & 0 \leq x < 2 \end{cases}$$

on $[-2, 2]$. Then, use Theorem 2.3 to determine the value of $F(x)$ on $[-2, 2]$.

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.