Differential Equations

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1 Daily Quiz

2 Key Topics

Today, we discuss the Fourier series of $f(x)$ on $[-L, L]$ when $f(x)$ is an even or an odd function. In addition, we introduce the Fourier Cosine and Sine series. For further reading, see [1, Sections 11.2–11.3].

Recall that we denote the Fourier series of f on $[-L, L]$ by

$$F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right), \tag{1}$$

where $L > 0$. Also, the coefficients of the Fourier series in (1) can be computed as follows:

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi}{L}x\right)dx,\ m \geq 0$$
$$b_m = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)dx,\ m \geq 1$$

2.1 Examples

Example 2.1. Recall that the Fourier series of

$$f(x) = \begin{cases} -x & -2 \leq x < 0 \\ x & 0 \leq x < 2 \end{cases} \tag{2}$$

on the interval $[-2, 2]$ is given by

$$F(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi}{2}x\right)$$

Note that only cosine terms appear in the Fourier series, this is due to the fact that $f(x)$ is an even function.

Example 2.2. Recall that the Fourier series of $f(x) = x$ on the interval $[-2, 2]$ is given by

$$F(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}x\right)$$

Note that only sine terms appear in the Fourier series, this is due to the fact that $f(x)$ is an odd function.

Example 2.3. Recall that the Fourier series of

$$f(x) = \begin{cases} -x & -2 \leq x < 0 \\ \frac{1}{2} & 0 \leq x < 2 \end{cases} \tag{3}$$

on the interval $[-2, 2]$ is given by

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(n\pi) - 1}{n^2} \cos\left(\frac{n\pi}{2}x\right) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1 + 3\cos(n\pi)}{n} \sin\left(\frac{n\pi}{2}x\right)$$

Note that both cosine and sine terms appear in the Fourier series, this is due to the fact that $f(x)$ is neither even nor odd.

2.2 Even and Odd Functions

Let f be a function on $[-L, L]$. We say that f is an *even function* if $f(-x) = f(x)$ for all $x \in [-L, L]$. We say that f is an *odd function* if $f(-x) = -f(x)$ for all $x \in [-L, L]$. Note that

- the product of two even functions is even,
- the product of two odd functions is even,
- the product of an even and an an odd function is odd.

Theorem 2.4. *If f is an even function on $[-L, L]$, then its Fourier series on $[-L, L]$ has coefficients $b_m = 0$ for all $m \geq 1$ and*

$$a_m = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx, \ \ m \geq 0.$$

If f is an odd function on $[-L, L]$, then its Fourier series on $[-L, L]$ has coefficients $a_m = 0$ for all $m \geq 0$ and

$$b_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx, \ \ m \geq 1.$$

2.3 Fourier Cosine and Sine Series

Let f be a function on $[0, L]$. Define

$$f_e(x) = \begin{cases} f(-x) & -L \leq x < 0 \\ f(x) & 0 \leq x \leq L \end{cases}.$$

Then, f_e is an even extension of f. Therefore, by Theorem 2.4, the Fourier series of f_e on $[-L, L]$ has the form

$$F_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right),$$

where

$$a_m = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{m\pi}{L}x\right) dx, \ \ m \geq 0.$$

We denote by $C(x)$ the *Fourier Cosine* series of f on $[0, L]$, which is defined by

$$C(x) = F_e(x), \ \ x \in [0, L].$$

Theorem 2.5. *If f and f' are piecewise continuous on $[0, L]$, then $C(x)$ converges. Moreover,*

- $C(0) = f(0^+)$
- $C(L) = f(L^-)$
- *If $0 < x < L$ and f is continuous at x, then $C(x) = f(x)$*
- *If $0 < x < L$ and f is discontinuous at x, then $C(x) = \frac{f(x^-)+f(x^+)}{2}$*

Similarly, the *Fourier Sine* series of f on $[0, L]$ is defined by

$$S(x) = F_o(x),\ x \in [0, L],$$

where $F_o(x)$ is the Fourier series of the odd extension of f:

$$f_o(x) = \begin{cases} -f(-x) & -L \le x < 0 \\ f(x) & 0 \le x \le L \end{cases}.$$

Theorem 2.6. *If f and f' are piecewise continuous on $[0, L]$, then $S(x)$ converges. Moreover,*

- $S(0) = 0$
- $S(L) = 0$
- *If $0 < x < L$ and f is continuous at x, then $S(x) = f(x)$*
- *If $0 < x < L$ and f is discontinuous at x, then $S(x) = \frac{f(x^-)+f(x^+)}{2}$*

3 Exercises

Let $f(x) = 1 + x^2$ on $[0, 2]$

a. Find the Fourier Cosine series of f on $[0, 2]$.

b. Find the Fourier Sine series of f on $[0, 2]$.

c. Compare the Fourier Cosine and Sine series of f on $[0, 2]$.

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.