Differential Equations

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1 Daily Quiz

Draw the even and odd extension of $f(x) = 1 + x^2$ on $[0, 2]$.

2 Key Topics

Today, we finish our discussion of Fourier Cosine and Sine series. In addition, we introduce the method of separation of variables to solve the heat equation:

$$u_t = \alpha^2 u_{xx},\ 0 \leq x \leq L,\ t > 0, \tag{1}$$

which is a partial differential equation that describes the temperature $u(x, t)$ of an insulated rod of length L, as shown in Figure 1. For further reading, see [1, Sections 11.3 and 12.1].

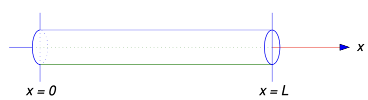

Figure 1: Insulated rod of length L

2.1 Fourier Cosine and Sine series

Let $f(x) = 1 + x^2$ on $[0, 2]$. Then, the Fourier Cosine series of f on $[0, 2]$ is given by

$$C(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}x\right),$$

where

$$\begin{aligned} a_0 &= \frac{14}{3} \\ a_n &= \int_0^2 (1 + x^2) \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \frac{16}{n^2\pi^2} \cos(n\pi),\ n \geq 1. \end{aligned}$$

Therefore,

$$C(x) = \frac{7}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi}{2}x\right)$$

Similarly, the Fourier Sine series on f on $[0, 2]$ is given by

$$S(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}x\right),$$

where

$$
\begin{aligned}
b_n &= \int_0^2 \left(1+x^2\right) \sin\left(\frac{n\pi}{2}x\right) dx \\
&= -\frac{2}{n^3\pi^3}\left(8-\pi^2 n^2+\left(5\pi^2 n^2-8\right)\cos(n\pi)\right)
\end{aligned}
$$

Therefore,

$$
S(x) = -\frac{2}{\pi^3}\sum_{n=1}^{\infty} \frac{8-\pi^2 n^2+\left(5\pi^2 n^2-8\right)\cos(n\pi)}{n^3}\sin\left(\frac{n\pi}{2}x\right)
$$

2.2 Separation of Variables

The heat equation in (1) will come with conditions on the initial temperature distribution and the boundary of the rod in Figure 1. For example, the conditions in (2) state that the ends of the rod are held at a constant temperature of 0 and the initial temperature distribution across the rod is given by $f(x)$.

$$
\begin{aligned}
u(0,t) = 0,\ u(L,t) = 0,\ t > 0 \\
u(x,0) = f(x),\ 0 \le x \le L
\end{aligned} \tag{2}
$$

The method of separation of variables assumes that $u(x,t) = X(x)T(t)$. Then, the heat equation can be written as

$$
X(x)T'(t) = \alpha^2 X''(x)T(t),\ 0 \le x \le L,\ t > 0,
$$

which can further be written as

$$
\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)},\ 0 \le x \le L,\ t > 0.
$$

Since the left side of this equation depends only on t and the right side of this equation depends only on x, the above equation is true if and only if both sides are equal to the same constant, which we denote by $-\lambda$. Hence, the heat equation can be written as

$$
\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda,\ 0 \le x \le L,\ t > 0.
$$

Hence, the heat equation in (1) with the conditions in (2) can be decomposed into a boundary value problem

$$
\begin{aligned}
X''(x) + \lambda X(x) = 0 \\
X(0) = 0,\ X(L) = 0
\end{aligned} \tag{3}
$$

and the differential equation

$$
T'(t) + \alpha^2 \lambda T(t) = 0 \tag{4}
$$

The boundary value problem in (3) has eigenvalues and corresponding eigenfunctions

$$
\lambda_n = \frac{n^2\pi^2}{L^2},\ X_n(x) = \sin\left(\frac{n\pi}{L}x\right),\ n \ge 1.
$$

Plugging the eigenvalues into the differential equation in (4) gives

$$
T'(t) = -\alpha^2\lambda_n T(t) = -\frac{\alpha^2 n^2 \pi^2}{L^2}T(t),
$$

which has the following as a solution

$$
T_n(t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}.
$$

For each $n \geq 1$, define

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right), \end{aligned}$$

which is a solution of the heat equation in (1) and satisfies the conditions $u_n(0,t) = 0$ and $u_n(L,t) = 0$. Next, define

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \alpha_n u_n(x,t) \\ &= \sum_{n=1}^{\infty} \alpha_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right), \end{aligned}$$

where

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

so that $u(x,0)$ is the Fourier sine series of $f(x)$ on $[0,L]$.

3 Exercises

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.