Differential Equations

Thomas R. Cameron

November 3, 2023

1 Daily Quiz

2 Key Topics

Today, we continue examples on how power series can used to solve differential equations. Furthermore, we introduce theory that states when the solution of a differential equation can be expressed as a power series. For further reading, see [1, Sections 7.2].

2.1 Examples

Example 2.1. Consider the differential equation

$$y'' - xy = 0,$$

and assume that the solution can be written as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

with a positive radius of convergence. Then, all derivatives of y exist and can be attained with term-by-term differentiation. In particular,

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Also, note that

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n$$

Plugging the power series of xy and y'' into the differential equation gives

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n \\ &= 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1}\right] x^n \end{aligned}$$

Therefore, $a_2 = 0$ and $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$ It follows that

$$\begin{aligned} a_{3n} &= \frac{a_0}{(2\cdot 3)\cdot(5\cdot 6)\cdots((3n-1)\cdot(3n))} \\ a_{3n+1} &= \frac{a_1}{(3\cdot 4)\cdot(6\cdot 7)\cdots((3n)\cdot(3n+1))} \\ a_{3n+2} &= 0, \end{aligned}$$

for $n \geq 1$, where a_0 and a_1 are arbitrary constants. Therefore, the solution to the differential equation can be written as

$$y = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1) \cdot (3n)} \right) + a_1 \left(x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n) \cdot (3n+1)} \right)$$

2.2 Theory

Thus, far we've always assume that we could represent the solution of a differential equation as a power serious. The following theorem gives us sufficient conditions for when this assumption is appropriate.

Theorem 2.2. *Consider the differential equation*

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0,$$

where $p_0(x)$, $p_1(x)$, and $p_2(x)$ are polynomials with no common factors. Let x_0 be any real value such that $p_0(x_0) \neq 0$ and let $\rho > 0$ denote the distance from x_0 to the nearest zero of $p_0(x)$ (possibly in the complex plane). Note that $\rho = \infty$ if $p_0(x)$ is constant. Then, every solution of the given differential equation can be written as a power series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n ,$$

which converges on the interval $(x_0 - \rho, x_0 + \rho)$.

Example 2.3. Consider the differential equation

$$(1 - x^2)y'' - xy' + y = 0$$

By Theorem 2.2, the solution to this differential equation can be represented as a power series, centered at 0, that converges on the interval $(-1, 1)$. To this end, let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ y'' &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \end{aligned}$$

Then,

$$xy' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} = \sum_{n=1}^{\infty} n a_n x^n$$

and

$$\begin{aligned} (1 - x^2)y'' &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n \\ &= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} - n(n-1) a_n \right] x^n \end{aligned}$$

Therefore, the differential equation can be written as

$$\begin{aligned} 0 &= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= (2a_2 + a_0) + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n-1)(n+1)a_n\right] x^n \end{aligned}$$

Let a_0 and a_1 be arbitrary constants. Then, $a_2 = -a_0/2$, $a_3 = 0$, and

$$a_{n+2} = \frac{(n-1)(n+1)}{(n+2)(n+1)} a_n = \frac{n-1}{n+2} a_n,$$

for $n \geq 2$. So, $a_{2n+1} = 0$ for $n \geq 1$ and

$$\begin{aligned} a_4 &= \frac{1}{4} a_2 = -\frac{1}{4 \cdot 2} a_0 \\ a_6 &= \frac{3}{6} a_4 = -\frac{3 \cdot 1}{6 \cdot 4 \cdot 2} a_0 \\ a_8 &= \frac{5}{8} a_6 = -\frac{5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} a_0 \\ &\vdots \\ a_{2n} &= -\frac{(2n-3) \cdot (2n-5) \cdots (1)}{(2n)(2n-2) \cdots (2)} a_0 \end{aligned}$$

Therefore the power series solution can be written as

$$y = a_1 x + a_0 \left(1 - \frac{1}{2} x^2 - \sum_{n=2}^{\infty} \frac{(2n-3)(2n-5) \cdots (1)}{(2n)(2n-2) \cdots (2)} x^{2n}\right)$$

3 Exercises

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.