Differential Equations

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November 6, 2023

# 1 Daily Quiz

# 2 Key Topics

Today, we continue solving the differential equation

$$(1-x^2)y'' - xy' + y = 0.$$

Then, we see that some power series solutions lead to 2nd-order recurrence relations. For further reading, see [1, Sections 7.2].

## 2.1 1st-Order Recurrence Relations

Last time, we discussed the following theorem which provides sufficient conditions for when the general solution of a differential equation can be represented as a power series.

**Theorem 2.1.** *Consider the differential equation*

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0,$$

*where $p_0(x)$, $p_1(x)$, and $p_2(x)$ are polynomials with no common factors. Let $x_0$ be any real value such that $p_0(x_0) \neq 0$ and let $\rho > 0$ denote the distance from $x_0$ to the nearest zero of $p_0(x)$ (possibly in the complex plane). Note that $\rho = \infty$ if $p_0(x)$ is constant. Then, every solution of the given differential equation can be written as a power series*

$$y = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n,$$

*which converges on the interval $(x_0 - \rho, x_0 + \rho)$.*

Often, the power series representation can be found using a 1st-order recurrence relation as in the following example

*Example* 2.2. Consider the differential equation

$$(1-x^2)y'' - xy' + y = 0$$

By Theorem 2.1, the solution to this differential equation can be represented as a power series, centered at

0, that converges on the interval $(-1, 1)$. To this end, note that

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
xy' &= \sum_{n=1}^{\infty} n a_n x^n \\
(1-x^2)y'' &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^{n+2} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\
&= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] x^n
\end{aligned}$$

Therefore, the differential equation can be written as

$$\begin{aligned}
0 &= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= (2a_2 + a_0) + 6a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n-1)(n+1)a_n\right] x^n
\end{aligned}$$

Let $a_0$ and $a_1$ be arbitrary constants. Then, $a_2 = -a_0/2$, $a_3 = 0$, and

$$a_{n+2} = \frac{(n-1)(n+1)}{(n+2)(n+1)} a_n = \frac{n-1}{n+2} a_n,$$

for $n \geq 2$. So, $a_{2n+1} = 0$ for $n \geq 1$ and

$$\begin{aligned}
a_4 &= \frac{1}{4} a_2 = -\frac{1}{4 \cdot 2} a_0 \\
a_6 &= \frac{3}{6} a_4 = -\frac{3 \cdot 1}{6 \cdot 4 \cdot 2} a_0 \\
a_8 &= \frac{5}{8} a_6 = -\frac{5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} a_0 \\
&\vdots \\
a_{2n} &= -\frac{(2n-3) \cdot (2n-5) \cdots (1)}{(2n)(2n-2) \cdots (2)} a_0
\end{aligned}$$

Therefore the power series solution can be written as

$$y = a_1 x + a_0 \left(1 - \frac{1}{2}x^2 - \sum_{n=2}^{\infty} \frac{(2n-3)(2n-5)\cdots(1)}{(2n)(2n-2)\cdots(2)} x^{2n}\right)$$

## 2.2 2nd-Order Recurrence Relations

In the following example, we need a 2nd-order recurrence relation to find the power series representation of the solution of the given initial value problem. We provide the initial conditions so that the numerical values of the first 5 coefficients in the power series can be found.

*Example* 2.3. Consider the initial value problem

$$y'' + 3xy' + (4 + 2x^2)y = 0, \;\; y(0) = 2, \;\; y'(0) = -3.$$

By Theorem 2.1, the solution to this differential equation can be represented as a power series, centered at 0, that converges on $\mathbb{R}$. To this end, note that

$$\begin{aligned}
y'' &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \\
3xy' &= \sum_{n=1}^{\infty} 3na_nx^n \\
(4+2x^2)y &= 4\sum_{n=0}^{\infty} a_nx^n + 2x^2\sum_{n=0}^{\infty} a_nx^n \\
&= \sum_{n=0}^{\infty} 4a_nx^n + \sum_{n=2}^{\infty} 2a_{n-2}x^n \\
&= 4a_0 + 4a_1x + \sum_{n=2}^{\infty}\left[4a_n + 2a_{n-2}\right]x^n
\end{aligned}$$

Therefore, the differential equation can be written as

$$\begin{aligned}
0 &= 4a_0 + 4a_1x + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 3na_nx^n + \sum_{n=2}^{\infty}\left[4a_n + 2a_{n-2}\right]x^n \\
&= (4a_0 + 2a_2) + (7a_1 + 6a_3)\,x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (3n+4)a_n + 2a_{n-2}\right]x^n
\end{aligned}$$

Let $a_0$ and $a_1$ be arbitrary constants. Then, $a_2 = -2a_0$, $a_3 = -\frac{7}{6}a_1$, and

$$a_{n+2} = -\frac{(3n+4)a_n + 2a_{n-2}}{(n+1)(n+2)},$$

for $n \geq 2$.

The initial conditions imply that $a_0 = 2$ and $a_1 = -3$. Therefore, we have

$$\begin{aligned}
a_2 &= -2a_0 = -4,\ a_3 = -\frac{7}{6}a_1 = \frac{7}{2} \\
a_4 &= -\frac{10a_2 + 2a_0}{4\cdot 3} = -\frac{-40+4}{4\cdot 3} = \frac{36}{4\cdot 3} \\
a_5 &= -\frac{13a_3 + 2a_1}{5\cdot 4} = -\frac{\frac{91}{2} - 6}{5\cdot 4} = -\frac{79}{5\cdot 4\cdot 2}
\end{aligned}$$

## 3 Exercises

## References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.