Differential Equations

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November 7, 2023

1 Daily Quiz

2 Key Topics

Today, we review the Laplace transform for solving the initial value problem

$$ay'' + by' + cy = f(t),\ y(0) = y_0,\ y'(0) = y'_0, \tag{1}$$

where $f(t)$ is a piecewise continuous function. For further reading, see [1, Sections 8.1–8.5].

Recall the definition of the Laplace transform

$$L(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

and the table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}(f(t))$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\cos(\omega t)$	$\frac{s}{\omega^2+s^2}$
$\sin(\omega t)$	$\frac{\omega}{\omega^2+s^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-w^2}$
$\sinh(\omega t)$	$\frac{w}{s^2-w^2}$
$e^{at}f(t)$	$F(s-a)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$

Table 1: Laplace Transforms

2.1 Laplace Transform of the IVP

First, we review the Laplace transform of the derivative. Recall that $y(t)$ is *exponentially bounded* if

$$|y(t)| \leq Me^{at},\ t \geq t_0$$

for some constants M, a, t_0.

Proposition 2.1. *Suppose that $y(t)$ is continuous and $y'(t)$ is piecewise continuous. If $y(t)$ is exponentially bounded, then the Laplace transform of $y'(t)$ exists for $s > a$ and*

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) = sY(s) - y(0),$$

where $Y(s)$ is the Laplace transform of $y(t)$.

Proof. Note that

$$
\begin{aligned}
\mathcal{L}(y') &= \int_0^\infty e^{-st} y'(t) dt \\
&= e^{-st} y(t) \Big|_0^\infty + s \int_0^\infty e^{-st} y(t) dt \\
&= s\mathcal{L}(y) - y(0).
\end{aligned}
$$

□

Proposition 2.2. *Suppose that $y(t)$ and $y'(t)$ are continuous and $y''(t)$ is piecewise continuous. If $y(t)$ and $y'(t)$ are exponentially bonded, then the Laplace transform of $y''(t)$ exists for $s > a$ and*

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2 Y(s) - sy(0) - y'(0),$$

where $Y(s)$ is the Laplace transform of $y(t)$.

Proof. Note that

$$
\begin{aligned}
\mathcal{L}(y'') &= \int_0^\infty e^{-st} y''(t) dt \\
&= e^{-st} y'(t) \Big|_0^\infty + s \int_0^\infty e^{-st} y'(t) dt \\
&= e^{-st} y'(t) \Big|_0^\infty + s\left(s\mathcal{L}(y) - y(0)\right) \\
&= s^2\mathcal{L}(y) - sy(0) - y'(0).
\end{aligned}
$$

□

Finally, we transform the entire initial value problem.

Theorem 2.3. *Suppose that the Laplace transform of y, y', y'', f exist. Then, the Laplace transform of (1) exists and can be written as*

$$Y(s)\left(as^2 + bs + c\right) = F(s) + (as + b)\, y(0) + ay'(0).$$

3 Exercises

Solve the following initial value problem

$$y'' - 2y' = \begin{cases} 4 & 0 \le t < 1 \\ 6 & t \ge 1 \end{cases}, \ y(0) = 2, \ y'(0) = 3.$$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.