Differential Equations

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1 Daily Quiz

2 Key Topics

Today, we review how the method of separation of variables can be used to break the heat equation into two differential equations, one of which is a boundary value problem. Then, we discuss the effect of changing the boundary conditions of the heat equation. For further reading, see [1, 12.1].

Recall that the heat equation:

$$u_t = \alpha^2 u_{xx},\ 0 \leq x \leq L,\ t > 0 \tag{1}$$
$$u(0,t) = 0,\ u(L,t) = 0,\ t > 0 \tag{2}$$
$$u(x,0) = f(x),\ 0 \leq x \leq L \tag{3}$$

describes the temperature of the rod shown in Figure 1 with initial temperature distribution given by (3) and boundary conditions given by (2).

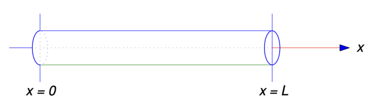

Figure 1: Insulated rod of length L

The method of separation of variables breaks the heat equation in (1) into two differential equations:

$$T' + \alpha^2 \lambda T = 0 \tag{4}$$

and

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) = 0,\ X(L) &= 0, \end{aligned} \tag{5}$$

the latter of which is a boundary value problem. The boundary value problem in (5) has eigenvalues and eigenfucntions:

$$\lambda_n = \frac{n^2\pi^2}{L^2},\ X_n = \sin\left(\frac{n\pi}{L}x\right),\ n = 1, 2, \ldots$$

For each eigenvalue λ_n, we have the following solution to (4):

$$T_n = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}.$$

Therefore, for $n = 1, 2, 3, \ldots$,

$$u_n(x,t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right)$$

satisfies both (1) and (2). So, the general solution

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n u_n(x,t) \\ &= \sum_{n=1}^{\infty} b_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi}{L} x\right) \end{aligned}$$

also satisfies both (1) and (2). In order for (3) to hold, we require

$$\begin{aligned} f(x) &= u(x,0) \\ &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right). \end{aligned}$$

Since $u(x,0)$ is a Fourier Sine series, it follows that

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

2.1 Different Boundary Conditions

Suppose that we change the boundary conditions in (2) to

$$u_x(0,t) = 0,\ u_x(0,L) = 0,\ t > 0. \tag{6}$$

Then, the boundary value problem in (5) becomes

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'(0) = 0,\ X'(L) &= 0, \end{aligned} \tag{7}$$

which has eigenvalues and eigenfunctions:

$$\lambda_n = \frac{n^2 \pi^2}{L^2},\ X_n = \cos\left(\frac{n\pi}{L} x\right),\ n = 0, 1, 2, \ldots.$$

Therefore, the general solution to the heat equation can be written as

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \cos\left(\frac{n\pi}{L} x\right).$$

Note that the general solution satisfies both (1) and (6). In order for (3) to hold, we require

$$\begin{aligned} f(x) &= u(x,0) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right). \end{aligned}$$

Since $u(x,0)$ is a Fourier Cosine series, it follows that

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right).$$

3 Exercises

Solve the following heat equations.

1.

$$\begin{aligned} u_t &= u_{xx},\ 0 \le x \le 2,\ t > 0 \\ u(0,t) &= 0,\ u(2,t) = 0,\ t > 0 \\ u(x,0) &= \begin{cases} x & 0 \le x < 1 \\ 2 - x & 1 \le x \le 2 \end{cases} \end{aligned}$$

2.

$$\begin{aligned} u_t &= u_{xx},\ 0 \le x \le 2,\ t > 0 \\ u_x(0,t) &= 0,\ u_x(2,t) = 0,\ t > 0 \\ u(x,0) &= e^x,\ 0 \le x \le 2. \end{aligned}$$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.