Differential Equations

Thomas R. Cameron

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1 Daily Quiz

2 Key Topics

Today, we introduce differential equations that model several real life phenomena. For more examples, see [2, Section 1.1] and [1, Section 1.1].

2.1 Population Growth and Decay

Let $P := P(t)$ represent the number of inhabitants in a given population as a function of time t. We assume that $P'(t)$ is proportional to $P(t)$, which gives the following differential equation

$$P'(t) = \alpha(P)P(t), \tag{1}$$

where $\alpha(P)$ is a continuous function of P. The Malthus model assumes that $\alpha(P)$ is constant, i.e., $\alpha(P) := \lambda$, which gives

$$P'(t) = \lambda P(t). \tag{2}$$

The Verhulst model assumes that $\alpha(P) := k(1 - \frac{1}{N}P)$, which gives

$$P'(t) = k(1 - \frac{1}{N}P(t))P(t). \tag{3}$$

2.2 Mass Spring System

Let $x := x(t)$ denote the position of an object attached by a spring to a wall. For simplicity, we assume that the object is lying on a flat frictionless surface and the spring satisfies Hooke's law, which gives us the following differential equation

$$mx''(t) = -kx(t), \tag{4}$$

where m denotes the mass of the object and k denotes the spring constant.

2.3 Predator-Prey System

Let $H := H(t)$ denote the number of Hare in a population and $L(t)$ denote the number of Lynx in a population. These two populations are dependent on eachother since the Lynx feed on the Hare. Moreover, under suitable assumptions, these two populations can be modeled by the following system of differential equations

$$\begin{aligned} H'(t) &= \alpha H(t) - \beta H(t)L(t), \\ L'(t) &= -\gamma L(t) + \delta H(t)L(t), \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ are constants.

3 Exercises

I. Verify that $P(t) = ce^{\lambda t}$ satisfies equation (2), for any constant c.

II. Verify that

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

satisfies equation (4), for any constants c_1, c_2.

III. A phase portrait of a potential solution to the predator-prey system is shown in Figure 1. Analyze this phase portrait: What happens as one population increases? What happens as one population decreases? Is there an equilibrium position?

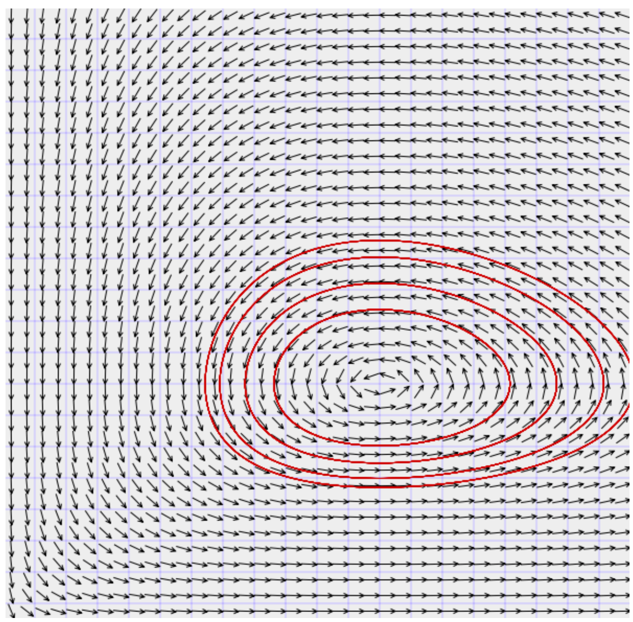

Figure 1: Phase portrait for predator-prey system

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.