Differential Equations

Thomas R. Cameron

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1 Daily Quiz

2 Key Topics

A *first-order differential equation* is any equation that can be written in the following form

$$F(t, y, y') = 0, \tag{1}$$

where $y := y(t)$ is a differentiable function of t. Any $y(t)$ that satisfies equation (1) is called a *solution* of the differential equation, any graph of $y(t)$ is called a *solution curve*.

A *first-order linear differential equation* is any first-order differential equation that can be written as

$$p(t)y' + q(t)y = f(t).$$

Any first-order differential equation that cannot be written in this form is said to be *non-linear*.

Today, we investigate *direction fields* for first-order differential equations that can be written as

$$y' = f(t, y),$$

i.e., the equation in (1) can be solved explicitly for y'. For further reading, see [1, Section 1.3] and [2, Section 1.3].

2.1 Direction Field

Suppose that $f(t, y)$ is defined on the rectangle

$$R = \{a \leq t \leq b,\ c \leq y \leq d\}.$$

Then, for every point $(t_0, y_0) \in R$, we know the value of $y'(t_0) = f(t_0, y_0)$, which is the slope of a solution curve at the point (t_0, y_0). Let

$$a = t_0 < t_1 < \cdots < t_m = b$$

be equally spaced values in $[a, b]$ and let

$$c = y_0 < y_1 < \cdots < y_n = d$$

be equally spaced values in $[c, d]$. Then, the points

$$(t_i, y_j),\ 0 \leq i \leq m,\ 0 \leq j \leq n$$

form a rectangular grid. Through each point on the grid, we draw a short line segment with slope $f(t_i, y_j)$, which results in a approximation of the direction field of the differential equation $y' = f(t, y)$.

Example 2.1. Consider the differential equation

$$y'(t) = \sin(t \cdot y). \tag{2}$$

We can approximate the direction field of the given differential equation over the rectangle

$$R = \{-\pi \le t \le \pi, \ \ -1 \le y \le 1\} \tag{3}$$

by creating the rectangular grid with

$$t_0 = -\pi, \ t_1 = -\pi/2, \ t_2 = 0, \ t_3 = \pi/2, \ t_4 = \pi$$

and

$$y_0 = -1, \ y_1 = -1/2, \ y_2 = 0, \ y_3 = 1/2, \ y_4 = 1.$$

Over each point in the rectangular grid, we draw a short line segment with slope $\sin(t_i \cdot y_j)$, sketch shown in class. □

Of course it is tedious to draw an approximate direction field by hand. Instead, we can use software to easily plot an accurate direction field. For example, Figure 1 shows the direction field of the differential equation in (2) over the rectangle in (3), with two solution curves.

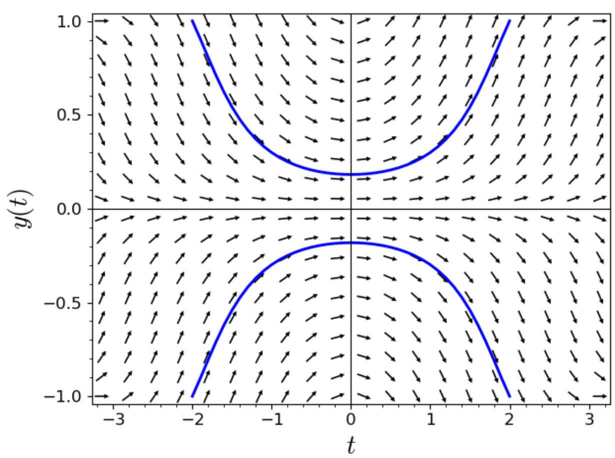

Figure 1: Direction field with two solution curves

3 Exercises

I. What solution curve of the differential equation in (2) goes through the origin?

II. Find the solution to the differential equation in (2) that satisfies $y(0) = 0$.

III. Is it possible for a solution curve to the differential equation in (2) to cross the t-axis? Why or why not?

IV. Perform Activity 1.3.1 in [1, Section 1.3].

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.