Differential Equations

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1 Daily Quiz

For each differential equation below, state its order and whether it is linear or non-linear:

a. $y''(t) + \sin(t)y(t) = \cos(t)$.

b. $2ty'(t) - 3y(t) = e^{-t}$.

c. $y'(t) + \frac{2}{y(t)} = 5$.

2 Key Topics

A *first-order initial value problem* can be written in the following form

$$F(t, y, y') = 0, \; y(t_0) = y_0 \tag{1}$$

where $y := y(t)$ is a differentiable function of t. Note that the first-order initial value problem is simply a first-order differential equation combined with an initial condition $y(t_0) = y_0$. This initial condition specifies exactly one solution curve, when a solution exists. Today, we investigate how to use the initial condition and differential equation to numerically approximate the solution. For further reading, see [1, Section 1.4] and [2, Chapter 3].

2.1 Euler's Method

Suppose that the differential equation can be solved explicitly for y', i.e., we have

$$y'(t) = f(t, y).$$

Assuming that $y(t)$ is twice differentiable, Taylor's theorem states that

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(c)}{2}(t - t_0)^2,$$

where $t > t_0$ and c lies between t_0 and t.

If $|t - t_0|$ is small, then the term $\frac{y''(c)}{2}(t - t_0)^2$ is negligible, which motivates the following approximation

$$y(t) \approx y(t_0) + y'(t_0)(t - t_0). \tag{2}$$

Define $h > 0$ to be a step-size and $t_1 = t_0 + h$. Furthermore, define $y_1 \approx y(t_1)$ via the equation in (2), i.e.,

$$y_1 = y_0 + hf(t_0, y_0).$$

In a similar manner, let $t_2 = t_1 + h$ and define the approximation $y_2 \approx y(t_2)$ as follows:

$$y_2 = y_1 + hf(t_1, y_1).$$

In general, we construct the equally spaced values $t_0 < t_1 < t_2 < \cdots < t_n$, where $t_i = t_{i-1} + h$, for $i = 1, \ldots, n$, and the approximations

$$y_i = y_{i-1} + hf(t_{i-1}, y_{i-1}).$$

3 Exercises

Consider the initial value problem

$$y'(t) = y + t,\ y(0) = 1.$$

I. Show that $y(t) = 2e^t - t - 1$ satisfies the given initial value problem.

II. Use Euler's method to approximate the solution to the given initial value problem at

$$t = 0,\ 0.1,\ 0.2,\ \ldots,\ 1.$$

III. Compare the exact value to the approximate values and describe what happens as t varies from 0 to 1.

IV. (*) Let $T > 0$ and approximate the solution to the given initial value problem over $[0, T]$ by splitting the interval into $n > 1$ sub-intervals and applying Euler's method. Show that

$$y_n = \left(1 + \frac{T}{n}\right)^n + \frac{T^2}{n^2} \cdot \frac{1}{\left(1 + \frac{T}{n}\right)^n} \cdot \sum_{k=1}^{n} \frac{k-1}{\left(1 + \frac{T}{n}\right)^k}$$

and conclude that

$$\lim_{n \to \infty} y_n = 2e^T - T - 1,$$

i.e., as $n \to \infty$ the approximations via Euler's method approach the exact solution.

Hint:

$$\frac{T^2}{n^2} \cdot \frac{1}{\left(1 + \frac{T}{n}\right)^n} \cdot \sum_{k=1}^{n} \frac{k-1}{\left(1 + \frac{T}{n}\right)^k} = \left(1 + \frac{T}{n}\right)^n - T - 1.$$

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.