Differential Equations

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1 Daily Quiz

Consider the initial value problem

$$y' = \cos(t \cdot y),\ \ y(0) = 1.$$

Find the approximation of $y(0.5)$ using a single step of Euler's method.

2 Key Topics

A first-order *autonomous differential equation* can be written in the following form

$$y' = f(y),$$

where $y := y(t)$ is a differentiable function of t. Note that an autonomous differential equation has a derivative that does not depend on the independent variable. Today we will investigate *one-parameter* families of differential equations, i.e., differential equations that can be written as follows

$$y' = f_\lambda(y).$$

In particular, we are interested in what impact changing the value of the parameter λ has on the solutions of the differential equation. Examples of one-parameter families include the Malthus model for population growth/decay:

$$P' = \lambda P.$$

For further reading, see [1, Section 1.7].

2.1 One-Parameter Families

Consider the one-parameter family

$$y' = y^2 - 4y + \lambda. \tag{1}$$

For $\lambda = 0$, the differential equation becomes

$$y' = y^2 - 4y = y(y - 4).$$

An *equilibrium solution* is any solution to a differential equation where $y' = 0$. Note that the above differential equation has two equilibrium solutions: $y = 0$ and $y = 4$. An equilibrium solution is *stable* if nearby solution curves converge toward it; conversely, an equilibrium solution is *unstable* if nearby solutions diverge away from it. As can be seen from Figure 1, the solution $y = 0$ is a stable equilibrium solution and the solution $y = 4$ is an unstable equilibrium solution. This conclusion could also be made by drawing a *phase line*, where the critical points are labeled and the sign of the derivative between critical points is indicated by an arrow (left for negative slope, right for positive slope). The phase line for the above differential equation was shown in class.

For $\lambda = 4$, the differential equation becomes

$$y' = y^2 - 4y + 4 = (y - 2)(y - 2),$$

which has a single equilibrium solution at $y = 2$. Moreover, this equilibrium point is *semi-stable* since some nearby solutions converge toward it and others diverse away from it, see Figure 1

For $\lambda = 8$, the differential equation becomes

$$y' = y^2 - 4y + 8,$$

which has no equilibrium solutions as can be seen in Figure 1.

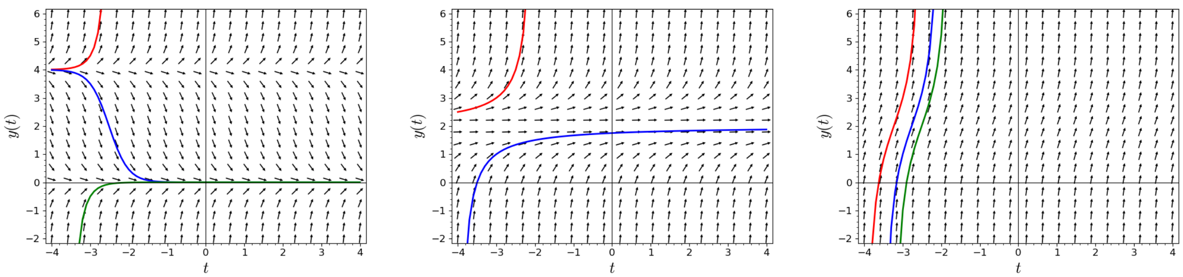

Figure 1: Direction field for $\lambda = 0, 4, 8$, respectively, with solution curves.

Note that the number of equilibrium solutions changes at $\lambda = 4$. We say that $\lambda = 4$ is a *bifurcation* value for the one-parameter family in (1). For $\lambda < 4$, we have two equilibrium solutions

$$y = 2 \pm \sqrt{4 - \lambda}.$$

When $\lambda = 4$ we have the single equilibrium solution $y = 2$. Finally, when $\lambda > 4$ we have no equilibrium solutions. We record all of this information in a bifurcation diagram, shown in Figure 2.

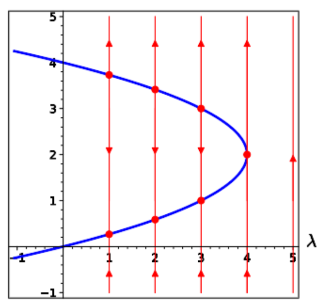

Figure 2: Bifurcation diagram for one-parameter family in (1).

3 Exercises

For each of the following one-parameter families, plot phase lines for

$$\lambda = -2, \ -1, \ 0, \ 1, \ 2$$

and find any bifurcation values. Combine all information into a single bifurcation diagram.

I. $y' = (1 - y)y + \lambda$

II. $y' = (\lambda - y^2)y$

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.