# Differential Equations

Thomas R. Cameron

August 28, 2023

## 1 Daily Quiz

Draw the phase line for the following differential equation

$$y' = y + 2.$$

## 2 Key Topics

A *linear first-order differential equation* can be written in the form

$$y' + p(t)y = q(t), \tag{1}$$

where $y := y(t)$ is a differentiable function of $t$. Today we will introduce the method of integration factor for solving linear first-order differential equations. For further reading, see [1, Section 1.5] and [2, Section 2.1].

### 2.1 Integration Factor

We seek a $\mu(t)$ such that

$$\mu(t)y' + \mu(t)p(t)y = \frac{d}{dt}F(t,y), \tag{2}$$

for some function $F(t,y)$. An equation in the form of (2) is called *exact*. Recall that the total derivative of $F(t,y)$ is given by

$$\frac{d}{dt}F(t,y) = \frac{\partial}{\partial t}F(t,y) + \frac{\partial}{\partial y}F(t,y)\frac{dy}{dt}. \tag{3}$$

By matching equations (2) and (3), we find that

$$\frac{\partial}{\partial y}F(t,y) = \mu(t).$$

Hence, $F(t,y) = \mu(t)y$. Furthermore,

$$\frac{\partial}{\partial t}F(t,y) = \mu'(t)y = \mu(t)p(t)y,$$

which implies that $\mu'(t) = \mu(t)p(t)$. Therefore,

$$\mu(t) = e^{P(t)}, \tag{4}$$

where $P(t)$ is any antiderivative of $p(t)$. The function $\mu(t)$ defined in (4) is known as an *integration factor* of the corresponding differential equation.

*Example* 2.1. Consider the linear first-order differential equation

$$y' - 3y = t.$$

Note that $P(t) = -3t$ is an anti-derivative of $p(t) = -3$. Hence, an integration factor is given by

$$\mu(t) = e^{-3t}.$$

## 3 General and Particular Solutions

Once an integration factor has been found for the linear first-order differential equation in (1), a *general solution* can be found as follows:

I. Multiply both sides of the differential equation by $\mu(t)$:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)q(t).$$

II. Re-write the left-hand side as an exact differential:

$$\frac{d}{dt}\mu(t)y = \mu(t)q(t).$$

III. Integrate both sides with respect to $t$:

$$\mu(t)y = \int \mu(t)q(t)dt$$

IV. Solve for $y$:

$$y = \frac{1}{\mu(t)} \int \mu(t)q(t)dt$$

Note that the indefinite integral has a "$+C$" constant built into its definitions. If an initial condition is given, then a *particular solution* can be formed by solving for the constant $C$.

*Example* 3.1. Consider the linear first-order initial value problem

$$y' - 3y = t,\ y(0) = 1.$$

We know that $\mu(t) = e^{-3t}$ is an integration factor. Hence,

$$y = e^{3t} \int te^{-3t}dt$$

is a general solution. Using integration by parts, we find

$$\begin{aligned}\int te^{-3t}dt &= -\frac{1}{3}te^{-3t} + \frac{1}{3}\int e^{-3t}dt \\ &= -\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + C.\end{aligned}$$

Therefore, the general solution is given by

$$\begin{aligned}y &= e^{3t}\left(-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + C\right) \\ &= -\frac{1}{3}t - \frac{1}{9} + Ce^{3t}\end{aligned}$$

By applying the initial condition, we have

$$y(0) = -\frac{1}{9} + C = 1,$$

which implies that $C = 1 + \frac{1}{9} = \frac{10}{9}$ and the particular solution is

$$y = -\frac{1}{3}t - \frac{1}{9} + \frac{10}{9}e^{3t}.$$

## 4 Exercises

Solve each of the following initial value problems.

I. $y' + \frac{1-t}{t} y = t^2,\ y(1) = 0$

II. $y' + y = \sin(t),\ y(0) = 1$

## References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.