Differential Equations

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1 Daily Quiz

Find the general solution of the differential equation

$$y' + \frac{1}{t}y = t^2.$$

2 Key Topics

A *separable first-order differential equation* can be written in the form

$$n(y)y' + m(t) = 0,$$

where $y := y(t)$ is a differentiable function of t. Today we will introduce the method of separation of variables to solve separable first-order differential equations. For further reading, see [1, Section 1.2] and [2, Section 2.2].

2.1 Separation of Variables

Every separable first-order differential equation can be written as an exact differential

$$\frac{d}{dt}\left(N(y) + M(t)\right) = 0,$$

where $\frac{d}{dy}N(y) = n(y)$ and $\frac{d}{dt}M(t) = m(t)$, i.e., $N(y)$ and $M(t)$ are antiderivatives of $n(y)$ and $m(t)$, respectively. Indeed, taking the derivative on the left hand side gives

$$N'(y)y' + M'(t) = 0 \Rightarrow n(y)y' + m(t) = 0.$$

So, the general solution to every separable first-order differential equation can be written as

$$N(y) + M(t) = C.$$

Therefore, every separable first-order differential equation can be solved by finding an antiderivative of $n(y)$ and $m(t)$.

Example 2.1. Consider the separable first-order differential equation

$$y' = \frac{t^2}{1 - y^2}.$$

We can rewrite the given differential equation as follows

$$(1 - y^2)y' - t^2 = 0,$$

hence it is separable. Note that

$$\frac{d}{dy}\left(y - \frac{1}{3}y^3\right) = 1 - y^2$$

and

$$\frac{d}{dt}\left(-\frac{1}{3}t^3\right) = -t^2.$$

Therefore, the general solution is given by

$$y - \frac{1}{3}y^3 - \frac{1}{3}t^3 = C.$$

□

3 Exercises

Solve each of the following initial value problems.

I. $y' = \sqrt{ty},\ y(1) = 1.$

II. $y' = \dfrac{3t^2 + 4t + 2}{2(y-1)},\ y(0) = -1.$

III. $y' = \dfrac{4t - t^3}{4 + y^3},\ y(0) = 1$

IV. $y' = \dfrac{t^2}{y(2 + t^3)},\ y(0) = -2.$

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.