Differential Equations

Thomas R. Cameron

September 11, 2023

1 Daily Quiz

For each DE below, state which method could be used to find a general solution:

1. $y' = ty + \sin(t)$
2. $y' = \dfrac{t}{y}$
3. $\left(t^2 y + y\right) y' + \left(ty^2 + t\right) = 0$

2 Key Topics

Today we introduce *first-order autonomous systems of differential equations* that can be written in the following form

$$x' = f(x, y),\ y' = g(x, y), \tag{1}$$

where $x := x(t)$ and $y := y(t)$ are differentiable functions. In particular, we will use direction fields to study the geometry of solution curves and Phase Plane analysis to study equilibrium solutions and their stability. For further reading see [1, Section 2.2].

3 Direction Field

Suppose that $f(x, y)$ and $g(x, y)$ are defined on the rectangle

$$R = \left\{(x, y) \colon\ a \leq x \leq b,\ c \leq y \leq d\right\}.$$

Then, for every point $(x_0, y_0) \in R$, we know the value of $x'(t)$ and $y'(t)$ (for all t since the system is autonomous). Let

$$a = x_0 < x_1 < \cdots < x_m = b$$

be equally spaced in $[a, b]$ and let

$$c = y_0 < y_1 < \cdots < y_m = d$$

be equally spaced in $[c, d]$. Then, the points

$$(x_i, y_j),\ 0 \leq i \leq m,\ 0 \leq j \leq n$$

form a rectangular grid. Through each point in the grid, draw a short line segment where the change in x is given by $f(x_i, y_j)$ and the change in y is given by $g(x_i, y_j)$, which results in a approximation of the direction field for the system in (1).

Example 3.1. Consider the first-order autonomous system

$$x' = x + y$$
$$y' = x - y$$

We can approximate the direction field over the rectangle

$$R = \{(x, y): \ -1 \leq x \leq 1, \ -1 \leq y \leq 1\}$$

by creating a rectangular grid with

$$x_0 = -1, \ x_1 = 0, \ x_2 = 1,$$

and

$$y_0 = -1, \ y_1 = 0, \ y_2 = 1,$$

At each point, we draw a short line segment where the change in x is given by $f(x_i, y_j)$ and the change in y is given by $g(x_i, y_j)$ (sketch shown in class).

4 Phase Plane

The *phase plane* for the system in (1) is constructed in the xy-plane as follows

I. Draw the *x-nullclines*, i.e., where $f(x, y) = 0$. The x-nullclines indicate where the solution curves will be vertical.

II. Draw the *y-nullclines*, i.e., where $g(x, y) = 0$. The y-nullclines indicate where the solution curves will be horizontal.

III. Draw the *equilibrium points*, i.e., where both $f(x, y) = 0$ and $g(x, y) = 0$.

IV. Label the regions where $x' > 0$ and where $x' < 0$, the regions are separated by x-nullclines. Similarly, label the regions where $y' > 0$ and where $y' < 0$.

V. Draw solution curves on and between each nullclines.

5 Exercises

Draw the phase plane for each of the following systems

I.

$$x' = x + y$$
$$y' = x - y$$

II.

$$x' = y - x^2$$
$$y' = x - 2$$

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.