Differential Equations

Thomas R. Cameron

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1 Daily Quiz

Find the general solution of the differential equation

$$ty' - y^2 = 0.$$

2 Key Topics

Today we consider first-order differential equations that can be written in the following form

$$n(t, y)y' + m(t, y) = 0. \tag{1}$$

This differential equation is *exact* if it can be rewritten as follows

$$\frac{d}{dt}F(t, y) = 0, \tag{2}$$

where $F(t, y)$ is a function that satisfies

$$\frac{\partial}{\partial t}F(t, y) = m(t, y) \text{ and } \frac{\partial}{\partial y}F(t, y) = n(t, y). \tag{3}$$

Then, the general solution of a first-order exact differential equation is given by

$$F(t, y) = C. \tag{4}$$

For further reading, see [1, Section 2.5].

2.1 Exact Differential Equations

Theorem 2.1. *Suppose that the functions $n(t, y)$ and $m(t, y)$ and their partial derivatives are continuous. Then, the differential equation in* (1) *is exact if and only if*

$$\frac{\partial}{\partial y}m(t, y) = \frac{\partial}{\partial t}n(t, y). \tag{5}$$

Proof. Suppose that the differential equation in (1) is exact. Then, there exists a function $F(t, y)$ that satisfies (3). Therefore, the mixed partials of $F(t, y)$ satisfy

$$\frac{\partial^2}{\partial y \partial t}F(t, y) = \frac{\partial}{\partial y}m(t, y) \text{ and } \frac{\partial^2}{\partial t \partial y}F(t, y) = \frac{\partial}{\partial t}n(t, y).$$

Since these mixed partial are continuous, they must be equal. Therefore, (5) holds.

Conversely, suppose (5) holds. We will construct $F(t, y)$ so that (3) holds. First, define $M(t, y)$ so that

$$\frac{\partial}{\partial t}M(t, y) = m(t, y).$$

Then, define $F(t,y) = M(t,y) + h(y)$. Since (3) must hold, it follows that

$$h'(y) = n(t,y) - \frac{\partial}{\partial y} M(t,y).$$

Hence, we can solve for $h(y)$ as long as the above equation does not depend on t, despite its appearance. Differentiating with respect to t shows that this is in fact the case:

$$\begin{aligned}\frac{\partial}{\partial t} n(t,y) - \frac{\partial^2}{\partial t \partial y} M(t,y) &= \frac{\partial}{\partial t} n(t,y) - \frac{\partial^2}{\partial y \partial t} M(t,y) \\ &= \frac{\partial}{\partial t} n(t,y) - \frac{\partial}{\partial y} m(t,y) = 0.\end{aligned}$$

□

3 Exercises

Check whether each of the following differential equations below is exact. If so, find the general solution.

a. $2t + y^2 + 2tyy' = 0$

b. $(y\cos(t) + 2te^y) + \left(\sin(t) + t^2 e^y - 1\right) y' = 0$

c. $\left(3ty + y^2\right) + \left(t^2 + ty\right) y' = 0$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.