Differential Equations

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1 Daily Quiz

Find a equilibrium solution for the first-order autonomous linear system:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

where a, b, c, d are constants.

2 Key Topics

Today we discuss analytic solutions to *first-order linear autonomous systems of differential equations* that can be written in the following form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

where $x := x(t)$ and $y := y(t)$ are differentiable functions. In particular, we will use eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

to form the general solution of the system in (1) and draw its phase plane. For further reading, see [1, Section 3.3].

3 Eigenvalues and Eigenvectors

A scalar λ and corresponding vector $\mathbf{v}$ is an *eigenvalue* and *eigenvector*, respectively, of a matrix A if

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Given a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the eigenvalues of A are given by the roots of the quadratic

$$\lambda^2 - \lambda\,(a + d) + (ad - bc)$$

Note that the *trace* of the matrix A is $(a + d)$ and the *determinant* of the matrix A is $(ad - bc)$.

Given the eigenvalues λ_1, λ_2, a corresponding eigenvector $\mathbf{v}_1$ can be formed from the null space of $\lambda_1 I - A$, and a corresponding eigenvector $\mathbf{v}_2$ can be formed from the null space of $\lambda_2 I - A$.

Example 3.1. Consider the first-order autonomous system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 18 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The eigenvalues are given by the roots of the quadratic

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

Hence, we have eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$. Furthermore, corresponding eigenvectors can be found from the null spaces

$$\lambda_1 I - A = \begin{bmatrix} -9 & 3 \\ -18 & 6 \end{bmatrix} \rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and

$$\lambda_2 I - A = \begin{bmatrix} -6 & 3 \\ -18 & 9 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

4 General Solution

Given the eigenvalues and corresponding eigenvectors of the matrix in (2), the general solution of the system in (1) can be written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t},$$

provided the eigenvalues λ_1 and λ_2 are real and distinct (this is the only case we will consider).

Example 4.1. The system in Example 3.1 has the following general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}.$$

The phase plane of the system in (1) can be formed from the general solution as follows:

I. Sketch the *stable line solutions*, i.e., the solution curves that are along the line spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

II. Draw arrows on the stable line solutions: point toward the origin if the corresponding eigenvalue is negative, point away from the origin if the corresponding eigenvalue is positive.

III. Sketch solution curves between each stable line solution.

IV. Classify the equilibrium solution at the origin:

- *Saddle*: if $\lambda_1 < 0 < \lambda_2$,
- *Sink*: if $\lambda_1 < \lambda_2 < 0$,
- *Source*: if $0 < \lambda_1 < \lambda_2$.

5 Exercises

Sketch the phase plane for the system in Example 3.1.

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.