Differential Equations

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September 18, 2023

1 Daily Quiz

2 Key Topics

Today we begin an investigation of higher order differential equations. In particular, we start with second-order linear homogeneous differential equations, which can be written in the following form

$$y'' + p(t)y' + q(t) = 0, \tag{1}$$

where $y := y(t)$ is a differentiable function of t. For further reading, see [1, Section 5.1].

2.1 Existence and Uniqueness

Theorem 2.1. *Suppose $p(t)$ and $q(t)$ are continuous on an open interval (a, b). Then, the initial value problem*

$$y'' + p(t)y' + q(t) = 0,\ y(t_0) = y_0,\ y'(t_0) = y'_0$$

has a unique solution for all $t_0 \in (a, b)$.

Example 2.2. Consider the differential equation

$$y'' - y = 0.$$

Note that both $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ satisfy the given differential equation.

Example 2.3. Consider the differential equation

$$y'' + \omega^2 y = 0.$$

Note that both $y_1(t) = \cos(\omega t)$ and $y_2(t) = \sin(\omega t)$ satisfy the given differential equation.

Example 2.4. Consider the differential equation

$$t^2 y'' + ty' - 4y = 0.$$

Note that both $y_1(t) = t^2$ and $y_2(t) = 1/t^2$ satisfy the given differential equation.

2.2 General Solutions

Given $y_1(t)$ and $y_2(t)$ defined on an interval (a, b) and constant c_1 and c_2, the function

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{2}$$

is a *linear combination* of y_1 and y_2.

Theorem 2.5. *If y_1 and y_2 are solutions of the differential equation in* (1)*, then the linear combination in* (2) *is also a solution for any constants c_1 and c_2.*

Proof. Suppose that y_1 and y_2 are solutions of the differential equation in (1). Let c_1 and c_2 be constants and define $y(t)$ as the linear combination in (2). Then

$$\begin{aligned} y'' + p(t)y' + q(t)y &= (c_1 y_1'' + c_2 y_2'') + p(t)\,(c_1 y_1' + c_2 y_2') + q(t)\,(c_1 y_1 + c_2 y_2) \\ &= c_1\,(y_1'' + p(t)y_1' + q(t)y_1) + c_2\,(y_2'' + p(t)y_2' + q(t)y_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

□

We say that $\{y_1, y_2\}$ forms a *fundamental set of solutions* to (1) if every solution to (1) on can be written as a linear combination of y_1 and y_2. In this case, we say that the linear combination in (2) forms a *general solution* to (1).

We say that $\{y_1, y_2\}$ forms a *linearly independent set* if y_1 and y_2 are not constant multiples of each other.

Theorem 2.6. *Suppose $p(t)$ and $q(t)$ are continuous on (a, b). Then, $\{y_1, y_2\}$ forms a fundamental set of solutions to* (1) *if and only if $\{y_1, y_2\}$ is a linearly independent set.*

3 Exercises

I. Show that the the solutions y_1 and y_2 form a fundamental set for each differential equation in Examples 2.2–2.3. Then, construct the general solution for each differential equation.

II. Find the particular solution for each differential equation in Examples 2.2–2.3 with the additional initial conditions

$$y(0) = 1,\ y'(0) = 2.$$

III. Show that there are infinitely many solutions to the differential equation in Example 2.3 that satisfy the boundary conditions

$$y(0) = 1,\ y(\pi) = -1.$$

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.