Differential Equations

Thomas R. Cameron

September 19, 2023

1 Daily Quiz

Find the particular solution of the initial value problem:

$$y'' + 3y' + 2y = 0,\ y(0) = 1,\ y'(0) = 1,$$

given the fact that $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-t}$ are both solutions of the differential equaiton.

2 Key Topics

Today, we introduce the Wronskian for second-order linear homogeneous differential equations, which can be used to determine when $\{y_1, y_2\}$ forms a fundamental set of solutions. For further reading, see [1, Section 5.1].

2.1 The Wronskian

Let y_1 and y_2 be solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0, \tag{1}$$

where $y := y(t)$ is a differentiable function of t. The *Wronskian* of $\{y_1, y_2\}$ is defined as the determinant:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'. \tag{2}$$

If $p(t)$ and $q(t)$ are continuous on (a, b), then the initial value problem

$$y'' + p(t)y' + q(t)y = 0,\ y(t_0) = y_0,\ y'(t_0) = y_0', \tag{3}$$

has a unique solution for any $t_0 \in (a, b)$ and for any $y_0, y_0' \in \mathbb{R}$. In this case, $\{y_1, y_2\}$ forms a fundamental set if and only if

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0' \end{aligned} \tag{4}$$

has a unique solution for all $t_0 \in (a, b)$ and for all $y_0, y_0' \in \mathbb{R}$, which is true if and only if the Wronskian is non-zero, i.e., $W(y_1, y_2) \neq 0$.

2.2 Abel's Formula

Interestingly, the Wronskian can be expressed as a natural exponent. Indeed, by differentiating both sides of the determinant in (2) we find

$$W' = y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1'' = y_1y_2'' - y_2y_1''.$$

Since y_1 and y_2 are solutions to the differential equation in (1), it follows that

$$y_1'' = -p(t)y_1' - q(t)y_1 \text{ and } y_2'' = -p(t)y_2' - q(t)y_2$$

Therefore, the derivative of the Wronskian can be written as

$$\begin{aligned} W' &= -y_1\left(p(t)y_2' + q(t)y_2\right) + y_2\left(p(t)y_1' + q(t)y_1\right) \\ &= -p(t)\left(y_1y_2' - y_2y_1'\right) - q(t)\left(y_1y_2 - y_1y_2\right) \\ &= -p(t)W. \end{aligned}$$

So, the derivative of W is equal to a multiple of W; hence, we have the following general solution

$$W = Ce^{-\int p(t)dt}. \tag{5}$$

Equation 5 implies that the Wronskian is either always zero or never zero on any open interval where $p(t)$ is continuous. Therefore, we can check the Wronskian in (2) at a single value of t.

3 Exercises

Consider the differential equation:

$$y'' + 3y' + 2y = 0,$$

with the fundamental set of solutions $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-t}$.

I. Find the Wronskain using (2).

II. Verify that equation (5) holds for some value of C.

III. Verify that $\{y_1, y_2\}$ forms a fundamental set.

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.