Differential Equations

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1 Daily Quiz

Find the general form of the Wronksian associated with the differential equation

$$y'' + ty' + 2y = 0.$$

2 Key Topics

Today, we introduce a method for finding a fundamental set of solutions for second-order linear homogeneous differential equations of the form:

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, c are constants and $a \neq 0$. Such differential equations are said to have constant coefficients. For further reading, see [2, Section 5.2] or [1, Section 4.1]

2.1 The Characteristic Equation.

We seek solutions to (1) of the form $y = e^{rt}$. Plugging into the differential equation gives

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0.$$

Factoring out the e^{rt} term gives

$$e^{rt}\left(ar^2 + br + c\right) = 0.$$

Since e^{rt} is never zero, it follows that $\left(ar^2 + br + c\right) = 0$. Using the quadratic formula gives

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

If $b^2 - 4a > 0$, then we have two distinct real solutions to (2), which we denote by r_1, r_2. Define $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$. Then, the Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{(r_1 + r_2)t}\left(r_2 - r_1\right) \neq 0.$$

Hence $\{y_1, y_2\}$ forms a fundamental set for the differential equation in (1).

2.2 Repeated and Complex Roots

If $b^2 - 4ac = 0$, then we have a single repeated solution to (2), which we denote by r. We will show that $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ form a fundamental set for the differential equation in (1).

If $b^2 - 4ac < 0$, then we have two distinct complex solutions to (2), which we denote by $r_1 = w + iz$ and $r_2 = w - iz$. We will show that $y_1(t) = e^{wt}\cos(zt)$ and $y_2(t) = e^{wt}\sin(zt)$ form a fundamental set for the differential equation in (1).

3 Exercises

Solve the following initial value problems.

I. $y'' + 6y' + 5y = 0,\ y(0) = 3,\ y'(0) = -1.$

II. $y'' + 7y' + 12y = 0,\ y(1) = 3,\ y'(1) = 5.$

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.