Differential Equations

Thomas R. Cameron

September 22, 2023

1 Daily Quiz

Find the roots of the characteristic equation associated with the differential equation

$$y'' + 5y' + 6y = 0.$$

2 Key Topics

Today we discuss solutions for second-order linear homogeneous constant coefficient differential equations:

$$ay'' + by' + cy = 0,\ a \neq 0, \tag{1}$$

where the characteristic equation

$$ar^2 + br + c = 0$$

has a single repeated root r. For further reading, see [2, Section 5.2] or [1, Section 4.1]

2.1 Fundamental Set of Solutions

We can form a fundamental set of solutions in the case of a repeated root with

$$y_1(t) = e^{rt},\ y_2(t) = te^{rt}.$$

First, we must show that y_1 and y_2 satisfy the differential equation in (1). To this end, note that

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= a(r^2e^{rt}) + b(re^{rt}) + c(e^{rt}) \\ &= e^{rt}\left(ar^2 + br + c\right) = 0, \end{aligned}$$

where $ar^2 + br + c = 0$ since r is a root of the characteristic equation. Also,

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a\left(2re^{rt} + r^2te^{rt}\right) + b\left(e^{rt} + rte^{rt}\right) + c(te^{rt}) \\ &= te^{rt}\left(ar^2 + br + c\right) + e^{rt}\left(2ar + b\right) \\ &= 0 + 0, \end{aligned}$$

where $2ar + b = 0$ since the single repeated root is $r = -\frac{b}{2a}$.

Second, we must show that the Wronskian is non-zero. To this end, note that

$$W(y_1, y_2) = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & e^{rt} + rte^{rt} \end{vmatrix} = e^{2rt}.$$

3 Exercises

Solve the following initial value problems.

I. $y'' + 6y' + 9y = 0,\ y(0) = 3,\ y'(0) = -1.$

II. $y'' + 4y' + 4y = 0,\ y(1) = 3,\ y'(1) = 5.$

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.