Differential Equations

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1 Daily Quiz

Find a fundamental set of solutions for the differential equation

$$y'' + 2y' + y = 0.$$

2 Key Topics

Today we discuss solutions for second-order linear homogeneous constant coefficient differential equations:

$$ay'' + by' + cy = 0, \; a \neq 0, \tag{1}$$

where the characteristic equation

$$ar^2 + br + c = 0$$

has two complex roots. For further reading, see [2, Section 5.2] or [1, Section 4.1].

2.1 Fundamental Set

Assuming that a, b, c are real constants, then the complex roots of the characteristic equation will come in complex conjugate pairs

$$r_1 = w + iz, \; r_2 = w - iz,$$

where $i = \sqrt{-1}$ and

$$w = -\frac{b}{2a}, \; z = \frac{\sqrt{4ac - b^2}}{2a}.$$

Since r_1 and r_2 are distinct, we can use

$$y_1(t) = e^{r_1 t}, \; y_2(t) = e^{r_2 t}$$

to form a fundamental set. However, it is not ideal to represent the solutions using complex functions.

Using Euler's formula, we can rewrite the fundamental set as follows

$$\begin{aligned} y_1(t) &= e^{wt} e^{izt} \\ &= e^{wt} \left(\cos(zt) + i \sin(zt) \right) \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= e^{wt} e^{-izt} \\ &= e^{wt} \left(\cos(-zt) + i \sin(-zt) \right) \\ &= e^{wt} \left(\cos(zt) - i \sin(zt) \right). \end{aligned}$$

Since $\{y_1, y_2\}$ form a fundamental set, any linear combination of these functions is also a solution to the differential equation in (1). Hence,

$$z_1(t) = \frac{y_1(t) + y_2(t)}{2} = e^{wt}\cos(zt)$$

and

$$z_2(t) = \frac{y_1(t) - y_2(t)}{2i} = e^{wt}\sin(zt)$$

are also solutions to the the differential equation in (1). Furthermore,

$$\begin{aligned} W(z_1, z_2) &= \begin{vmatrix} e^{wt}\cos(zt) & e^{wt}\sin(zt) \\ we^{wt}\cos(zt) - ze^{wt}\sin(zt) & we^{wt}\sin(zt) + ze^{wt}\cos(zt) \end{vmatrix} \\ &= ze^{wt} \neq 0. \end{aligned}$$

Hence, $\{z_1(t), z_2(t)\}$ forms a fundamental set of solutions to the differential equation in (1).

2.2 Harmonic Oscillators

Consider a cart on wheels of mass m connected to a wall by a spring with spring constant k. Suppose the mass is pulled from its equilibrium position and then released. The restoring force in the spring causes the mass to move back and forth about its equilibrium position. The friction constant b of the wheels determines the motion of the cart after the initial release.

Let $y(t)$ denote the distance of the cart from its equilibrium position. Then, the following differential equation describes the motion of the cart

$$my'' + by' + ky = 0$$

3 Exercises

Find the general solution for each of the following differential equations.

I. $y'' + y = 0$

II. $y'' + y' + y = 0$

III. $y'' + 2y' + y = 0$

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.