Differential Equations

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September 26, 2023

1 Daily Quiz

Find a fundamental set of solutions for the differential equation

$$y'' + 2y' + 2y = 0.$$

2 Key Topics

Today we discuss solutions for second-order linear non-homogeneous differential equations:

$$y'' + p(t)y' + q(t)y = f(t), \tag{1}$$

where $y := y(t)$ is a differentiable function. For further reading, see [2, Section 5.3] or [1, Section 4.2].

2.1 General Solution

We say that y_p is a *particular solution* of the non-homogeneous differential equation in (1) if

$$y_p'' + p(t)y_p' + q(t)y_p = f(t).$$

Moreover, there is a *complementary homogeneous differential equation*

$$y'' + p(t)y' + q(t)y = 0.$$

Theorem 2.1. *Suppose $p(t), q(t), f(t)$ are continuous on an interval (a, b). Let y_p be a particular solution on (a, b) and let $\{y_1, y_2\}$ be a fundamental set of solutions on (a, b) of the complementary homogeneous differential equation. Then y is a solution of (1) if and only if*

$$y = y_p + c_1y_1 + c_2y_2,$$

for some constants c_1 and c_2.

Proof. Suppose that

$$y = y_p + c_1y_1 + c_2y_2.$$

Then,

$$\begin{aligned} y'' + p(t)y' + q(t)y &= \left(y_p'' + c_1y_1'' + c_2y_2''\right) + p(t)\left(y_p' + c_1y_1' + c_2y_2'\right) + q(t)\left(y_p + c_1y_1 + c_2y_2\right) \\ &= \left(y_p'' + p(t)y_p' + q(t)y_p\right) + c_1\left(y_1'' + p(t)y_1' + q(t)y_1\right) + c_2\left(y_2'' + p(t)y_2' + q(t)y_2\right) \\ &= f(t) + c_1(0) + c_2(0) = f(t). \end{aligned}$$

Now, suppose that $y(t)$ is a solution of (1). Then, $y - y_p$ is a solution to the complementary homogeneous differential equation. Indeed,

$$\begin{aligned} \left(y'' - y_p''\right) + p(t)\left(y' - y_p'\right) + q(t)\left(y - y_p\right) &= \left(y'' + p(t)y' + q(t)y\right) - \left(y_p'' + p(t)y_p' + q(t)y_p\right) \\ &= f(t) - f(t) = 0. \end{aligned}$$

Therefore,

$$y - y_p = c_1y_1 + c_2y_2,$$

for some constants c_1 and c_2. □

2.2 Undetermined Coefficients

The method of undetermined coefficients makes an assumption about the initial form of a particular solution y_p. For example, consider the non-homogeneous differential equation below

$$y'' + y' + 2y = 3.$$

One may guess a particular solution of the form $y_p = c$, where c is is some constant. Plugging this y_p into the differential equation gives

$$2c = 3,$$

which implies that $y_p = 3/2$ is a particular solution of the given differential equation.

However, the following non-homogeneous differential equation is not so easy:

$$y'' + y' = 3.$$

This time, if we guess a particular solution of the form $y_p = c$, where c is some constant, then we get $0 = 3$. This contradiction occurs because $y_p = c$ is a solution to the complementary homogeneous differential equation $y'' + y' = 0$, which has characteristic equation $r^2 + r = 0$. When a particular solution is also a solution to the complementary homogeneous equation, we must multiply it by t until it is no longer a solution. In this case, we have $y_p = ct$. Plugging y_p into the differential equation gives us

$$c = 3.$$

3 Exercises

Use the method of undetermined coefficients to find a particular solution for each of the following differential equations.

I. $y'' + y = 2e^{-t}$

II. $y'' + 2y' + y = 2e^{-t}$

III. $y'' + y' + y = \cos(t)$.

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.