Differential Equations

Thomas R. Cameron

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1 Daily Quiz

Find a particular solution to the non-homogeneous differential equation

$$y'' + 2y' + 2y = 3t + 5.$$

2 Key Topics

Today we use the method of undetermined coefficients to solve second-order linear non-homogeneous differential equations:

$$y'' + p(t)y' + q(t)y = f(t), \tag{1}$$

where $y := y(t)$ is a differentiable function. Recall that every solution to (1) is of the form

$$y = y_p + c_1 y_1 + c_2 y_2,$$

where y_p is a particular solution to (1) and $\{y_1, y_2\}$ form a fundamental set for the complementary homogeneous differential equation.

The method of undetermined coefficients makes an assumption about the initial form of a particular solution. The table below provides a summary of the basic forms for $f(t)$ and the corresponding initial guess for y_p. Note that y_p must be plugged into (1) in order to determined the constants $A_0, \ldots, A_n$. Furthermore, the y_p form may be multiplied by a power of t so that the given form is not a solution to the complementary homogeneous differential equation.

$f(t)$	$y_p(t)$
$\sum_{i=0}^n a_i t^i$	$\sum_{i=0}^n A_i t^i$
$\left(\sum_{i=0}^n a_i t^i\right) e^{at}$	$\left(\sum_{i=0}^n A_i t^i\right) e^{at}$
$\left(\sum_{i=0}^n a_i t^i\right) e^{at}\cos(bt)$	$\left(\sum_{i=0}^n A_i t^i\right) e^{at}\cos(bt) + \left(\sum_{i=0}^n B_i t^i\right) e^{at}\sin(bt)$
$\left(\sum_{i=0}^n a_i t^i\right) e^{at}\sin(bt)$	$\left(\sum_{i=0}^n A_i t^i\right) e^{at}\cos(bt) + \left(\sum_{i=0}^n B_i t^i\right) e^{at}\sin(bt)$

For further reading, see [2, Section 5.3] or [1, Section 4.2].

3 Exercises

Find the general solution for each of the following differential equations.

I. $y'' - 2y' + y = 3t^2 + 4t + 1$.

II. $y'' - 3y' - 4y = 3e^{2t}$.

III. $y'' - 3y' + 2y = 3e^{2t}$

IV. $y'' + 2y' + 2y = 5e^t \cos(t)$.

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.