Differential Equations

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1 Daily Quiz

Find a particular solution to the non-homogeneous differential equation

$$y'' + 2y' + 2y = 3t + 5.$$

2 Key Topics

Today we introduce the method of variation of parameters to find a particular solution of the second-order linear non-homogeneous differential equation:

$$y'' + p(t)y' + q(t)y = f(t), \tag{1}$$

where $y := y(t)$ is a differentiable function. For further reading, see [2, Section 5.7] or [1, Section 4.2].

2.1 Variation of Parameters

Let $\{y_1, y_2\}$ form a fundamental set for the complementary homogeneous equation of (1). We seek a particular of the form

$$y_p = u_1y_1 + u_2y_2, \tag{2}$$

where $u_1 := u_1(t)$ and $u_2 := u_2(t)$ are differentiable functions. The fact that y_p must satisfy (1) gives us one condition on u_1 and u_2. In order to determine u_1 and u_2 uniquely, we need impose the following second condition

$$u_1'y_1 + u_2'y_2 = 0.$$

Then, differentiating (2) gives

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' = u_1y_1' + u_2y_2'. \tag{3}$$

Differentiating again gives

$$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' \tag{4}$$

Substituting (2), (3), and (4) into (1) gives

$$\left(u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'\right) + p(t)\left(u_1y_1' + u_2y_2'\right) + q(t)\left(u_1y_1 + u_2y_2\right) = f(t),$$

which we rewrite as follows

$$u_1\left(y_1' + p(t)y_1' + q(t)y_1\right) + u_2\left(y_2'' + p(t)y_2' + q(t)y_2\right) + \left(u_1'y_1' + u_2'y_2'\right) = f(t).$$

Since $\{y_1, y_2\}$ is a fundamental set for the complementary homogeneous differential equation, it follows that the terms next to u_1 and u_2 are both zero. Therefore, the functions u_1' and u_2' must satisfy the following

$$u_1'y_1 + u_2'y_2 = 0, \tag{5}$$
$$u_1'y_1' + u_2'y_2' = f(t). \tag{6}$$

We can solve this system of equations for u_1' and u_2' as follows

$$u_1' = -\frac{y_2 f(t)}{W(y_1, y_2)}, \tag{7}$$

$$u_2' = \frac{y_1 f(t)}{W(y_1, y_2)}. \tag{8}$$

By integrating the above equations, we can find a particular solution of (1) in the form of (2).

3 Exercises

Find the general solution for each of the following differential equations.

I. $y'' - 5y' + 6y = 2e^t$

II. $y'' + y = \tan(t)$

References

[1] T. W. Judson, *The Ordinary Differential Equations Project*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2023.

[2] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.