Differential Equations

Thomas R. Cameron

September 5, 2023

1 Daily Quiz

Multiply both sides of the following differential equation by the factor $\mu = t$:

$$(3ty + y^2) + (t^2 + ty)y' = 0$$

and show that the differential equation becomes exact.

2 Key Topics

Today we consider first-order differential equations that can be written in the following form

$$n(t, y)y' + m(t, y) = 0 \tag{1}$$

and integration factors $\mu := \mu(t, y)$ that transform the differential equation into an exact differential equation. For further reading, see [1, Section 2.6]

Once the differential equation in (1) is transformed to an exact equation, the general solution is found by constructing $F(t, y)$ such that the differential can be written as

$$\frac{d}{dt}F(t, y) = 0,$$

which has a general solution of $F(t, y) = C$.

Example 2.1. The differential equation below is exact

$$\left(3t^2y + ty^2\right) + \left(t^3 + t^2y\right)y' = 0.$$

Define

$$F(t, y) = \left(t^3y + \frac{1}{2}t^2y^2\right) + h(y)$$

and note that

$$\frac{\partial F(t, y)}{\partial y} = \left(t^3 + t^2y\right) + h'(y).$$

Therefore, $\frac{\partial F}{\partial y} = n(t, y)$ implies that $h'(y) = 0$. Hence, we can set $h(y) = 0$ and it follows that the general solution of the differential equation is given by

$$t^3y + \frac{1}{2}t^2y^2 = C.$$

□

2.1 Transforming to Exact Differential Equations

Recall the following theorem, which we proved on 9/1/2023.

Theorem 2.2. *Suppose that the functions $n(t, y)$ and $m(t, y)$ and their partial derivatives are continuous. Then, the differential equation in* (1) *is exact if and only if*

$$\frac{\partial}{\partial y} m(t, y) = \frac{\partial}{\partial t} n(t, y). \tag{2}$$

Now, multiply both sides of the differential equation in (1) by the factor μ:

$$\mu(t, y)n(t, y)y' + \mu(t, y)m(t, y) = 0.$$

Then, the resulting differential equation is exact if and only if

$$\frac{\partial}{\partial y}\mu(t, y)m(t, y) = \frac{\partial}{\partial t}\mu(t, y)n(t, y).$$

Taking the partial derivatives on both sides gives us

$$m(t, y)\frac{\partial \mu(t, y)}{\partial y} + \mu(t, y)\frac{\partial m(t, y)}{\partial y} = n(t, y)\frac{\partial \mu(t, y)}{\partial t} + \mu(t, y)\frac{\partial n(t, y)}{\partial t},$$

which is a partial differential equation with unknown function μ. If we denote the partial derivatives as subscripts, then the above partial differential equation can be written as

$$m\mu_y - n\mu_t = \mu\left(n_t - m_y\right). \tag{3}$$

Sometimes the factor μ can be written as a function of a single variable, either t or y, exclusively. If $\mu := \mu(t)$, then (3) becomes

$$\frac{d}{dt}\mu = \mu\left(\frac{m_y - n_t}{n}\right). \tag{4}$$

If $\mu := \mu(y)$, then (3) becomes

$$\frac{d}{dy}\mu = \mu\left(\frac{n_t - m_y}{m}\right). \tag{5}$$

3 Exercises

Transform each differential equation to exact with an integration factor $\mu := \mu(t)$ or $\mu := \mu(y)$. Then, find the general solution of the exact differential equation.

a. Use (4) or (5) to ind the factor μ for the following differential equation

$$(3ty + y^2) + (t^2 + ty)y' = 0.$$

b. Use (4) or (5) to ind the factor μ for the following differential equation

$$\left(3t^2y^2 + t^2y^3 + 1\right)y' + 2ty^3 = 0.$$

Then, find the general solution to this differential equation.

References

[1] W. Trench, *Elementary Differential Equations with Boundary Value Problems*, Creative Commons Attribution-Noncommercial-Share Alike, 1st ed., 2013.